Høsten 2016

FYS100 Fysikk
Exam Solutions

Exam number **must** be written on every sheet.

The problem set is composed of 4 problems, that all need to be solved for a full score.
The standard formula sheet for FYS100 Fysikk is part of this problem set.
The problems are also attached in Norwegian.

Hints:
- Don't panic! - Read all the problems through first. - Do the easy parts first. - Include units on your answers. - 3 significant figures in the results. - Don't plug in numbers until the end. - Don't panic!!

You **must** draw sketches illustrating the problems and their solution!

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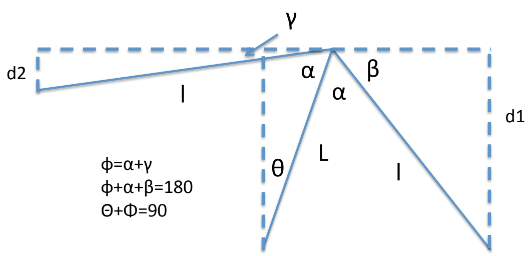
Below follows an example solution to the problems, as well as indications of the number of points for each problem. Sub-questions are approximately equally weighted within each problem (5-6-7-8 points for each) except 1A which gives more.

After each problem are listed typical mistakes, picked up during the correcting. While not exhaustive, for most students, these lists should provide a good starting point for matching their own solutions to the obtained grade.

Problem 1: Teeter-toy (20 points)



A teeter toy is composed of a massless central stick of length L and two massless sticks of length l attached at angles α, each with a mass m at the end (see the Figure). We imagine tilting the toy by an angle θ from the upright position.

a) Find an expression for the gravitational potential energy of the whole object, as a function of θ.

Solution: The potential energy of the two masses changes as they go up and down. And the geometry of the object determines how much one goes up, when the other goes down. The purpose is to find at which height the masses find themselves as a function of θ. Let us compute it relative to a zero point at the base of the short stick "L". Then for the lower and upper mass, respectively, we have (see figure)

$$U_1(\theta) = mg(L\cos\theta - d_1), \qquad U_1(\theta) = mg(L\cos\theta - d_2), \tag{1}$$

Now the trick is to find $d_{1,2}$, the vertical distances relative to the top point. $d_1 = l\sin\beta$, $d_2 = l\sin\gamma$.

Looking at the various triangles, we have $\theta + 90 + \phi = 180$. We also have $\phi + \alpha + \beta = 180$. And finally $\phi - \alpha = \gamma$. This gives

$$\beta = 90 - (\alpha - \theta), \qquad \gamma = 90 - (\alpha + \theta). \tag{2}$$

Remembering that $\sin(90 - x) = \cos(x)$, and that $\cos(x + y) = \cos x \cos y - \sin x \sin y$ we find

$$U(\theta) = U_1(\theta) + U_2(\theta) = 2mg\cos\theta(L - l\cos\alpha). \tag{3}$$

b) Show that $\theta = 0$ can be a stable equilibrium point, and find a criterion on L, l and α for this to be the case.

Solution : The first derivative of the potential is (minus) the force, and is

$$\frac{dU}{d\theta} = -2mg\sin\theta(L - l\cos\alpha), \tag{4}$$

which is zero when $\theta = 0$. That means that $\theta = 0$ is an equilibrium point. The second derivative gives us whether it is stable or not

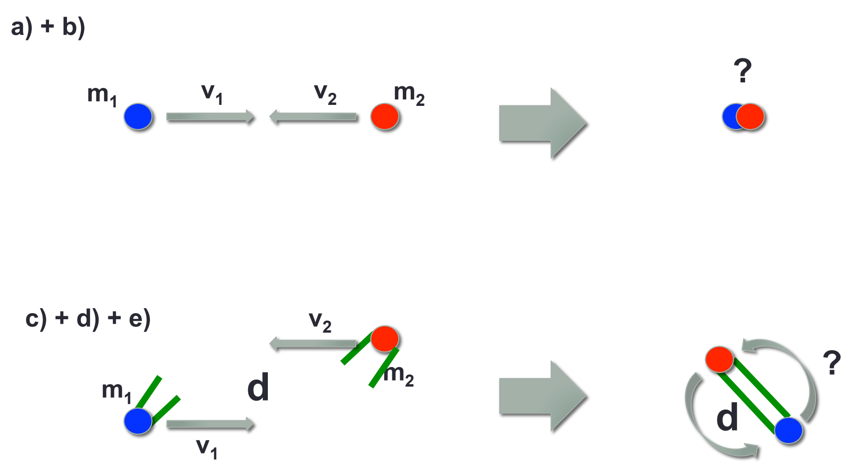
$$\frac{d^2U}{d\theta^2} = -2mg\cos\theta(L - l\cos\alpha), \tag{5}$$

which is positive (stable) at $\theta = 0$, if $L - l\cos\alpha < 0$. This happens whenever $l\cos\alpha > L$, so in case α is such that the two mass are below the support point in the equilibrium position. Otherwise, it is unstable.

For each question, provide an algebraic expression and a sketch.

Typical mistakes: This was a known hand-in problem, and many people simply knew it completely. But equally many had no clue, what to do. Many remembered the general idea, but got the geometry mixed up, or simply wrote down a guess of the function based on memory. Any honest attempt gave some partial credit. Many differentiated once and/or twice, but didn't explain (understand?) what those derivatives meant. Many also didn't evaluate the derivatives in $\theta = 0$, as required. 50% of the total available points were awarded overall.

Problem 2: Frost'y Collisions (30 points)



Elsa and Anna are skating on an icy lake, and approach each other from opposite directions with speeds v_1 and v_2. They have masses m_1 and m_2, and collide head-on, in such a way that their collision can be thought of as *perfectly inelastic (see Figure, top line). The surface of the lake can be taken as frictionless.*

a) What is their final speed v_f after the collision?

Solution: Momentum is conserved, and we find

$$m_1 v_1 - m_2 v_2 = (m_1 + m_2) v_f \to v_f = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2}. \quad (6)$$

Whether one defines v_2 to be positive or negative, will determine the sign in the numerator.

b) Is kinetic energy conserved? If not, how much did the kinetic energy change, and where did it go/come from?

Solution: Kinetic energy is not conserved, and explicitly

$$\begin{aligned} \Delta K &= \frac{1}{2}(m_1 + m_2) v_f^2 - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2 \\ &= -\frac{m_1 m_2}{2(m_1 + m_2)} (v_1 + v_2)^2, \end{aligned} \quad (7)$$

which is lost to internal energy.

The girls now skate apart again, turn around and perform a trick where they pass at a distance d, but so that they just manage to grab hold of each others hands (se Figure, bottom line). As a result, they start spinning around.

c) Where is their joint center of mass, as they collide (so when they are right "next to" each other, and their hands connect)? What is their combined moment of inertia around the center of mass I_{CM}?

Solution: Put the origin at the initial position of m_1 and define the initial position of m_2 to be (x_0, d). Then the CM at any time is at

$$r_{\mathrm{CM}} = \frac{m_2}{m_1 + m_2}(x_0, d) + \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} t(1, 0). \quad (8)$$

At "collision", it is just the y-component $r_{\mathrm{CM}} = m_2 d/(m_1 + m_2)$. One could have chosen the origin of the coordinate system to be halfway between the girls, shifting the y-coordinate by $d/2$. That's ok.
The moment of inertia relative to this CM is in any case

$$I_{\mathrm{CM}} = m_1 r_{\mathrm{CM}}^2 + m_2 (d - r_{\mathrm{CM}})^2 = \frac{m_1 m_2}{m_1 + m_2} d^2. \quad (9)$$

d) After the collision, what is the speed of the center of mass v_{CM}? What is the angular speed of rotation ω around the center of mass?

Solution: The speed of the CM follows from momentum conservation or just by differentiation of r_{CM}. Either way, it is as before

$$m_1 v_1 - m_2 v_2 = (m_1 + m_2) v_f \to v_f = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2}. \tag{10}$$

Angular momentum conservation around the CM (for instance in the CM frame) gives

$$m_1(v_1 - v_{\mathrm{CM}}) r_{\mathrm{CM}} + m_2(v_2 + v_{\mathrm{CM}})(d - r_{\mathrm{CM}}) = \frac{m_1 m_2}{(m_1 + m_2)} d(v_1 + v_2) = I_{\mathrm{CM}}\omega. \tag{11}$$

That means that $\omega = (v_1 + v_2)/d$.

e) Is kinetic energy conserved? If not, how much did the kinetic energy change, and where did it go/come from?

Solution: Let us calculate the difference between before and after, inserting our solutions from d), to find:

$$\frac{1}{2}(m_1 + m_2) v_{\mathrm{CM}}^2 + \frac{1}{2} I_{\mathrm{CM}}\omega^2 - \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_2 v_2^2 = 0 \tag{12}$$

Kinetic energy is conserved.

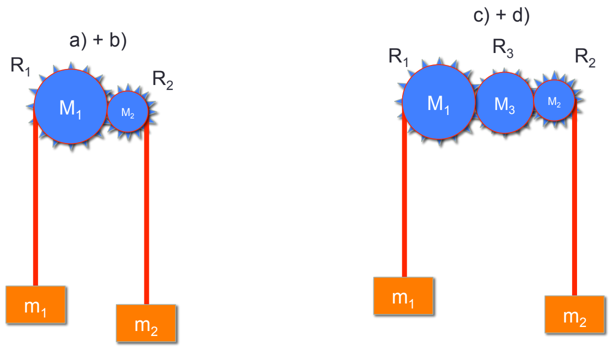
f) Extra Bonus-question [Joke-Warning!]: Do you want to build a snow-man?

Solution : Many inventive and surprisingly pop-culture-referential suggestions. No points awarded, sadly...

For each question, provide a sketch and an algebraic expression.

Typical mistakes: Momentum conservation went well in a). But in b) most of you just stated Yes or No, or wrote down Before and After but without inserting the result from a). Partial credit for that. In c), the majority assumed that the CM was halfway between the two girls. But that is only true if they have the same mass. Some of you got this wrong result by computing the CM wrong, taking both girls to have distance $d/2$, whereas one should have had $-d/2$. That propagated to the moment of inertia. In d), many used energy conservation, but very few used angular momentum conservation. And then things go wrong. In e), it was again mostly guesswork, as to whether the kinetic energy was conserved. f) was great! 43% of the total available points were awarded overall.

Problem 3: Cog Wheels (28 points)



Two cog wheels connect with each other, and each have a rope wound around them. Two blocks of masses $m_1 = 4.00$ kg and $m_2 = 3.00$ kg hang at the end of each rope. The cog wheels/pulleys have mass $M_1 = 2.00$ kg and radius $R_1 = 0.400$ m and mass $M_2 = 5.00$ kg and radius $R_2 = 0.300$ m, respectively, and can be thought of as uniform discs, $I^i_{\rm CM} = M_i R_i^2/2$. The ropes unwind without sliding. There is gravity $g = 9.80$ m/s^2. See Figure (left).

a) Draw a sketch with all the relevant forces and torques of the problem. What are the relations between the various accelerations and angular accelerations in the problem (the no-slipping conditions)? Think carefully about the force between the two cog wheels/pulleys.

Solution: There is gravity on each block $m_1 g$, $m_2 g$, there are string tensions T_1 and T_2, and then there is a force f between the pulleys, which helps one string force and hampers the other (depending on which one would otherwise rotate fastest). The no-slipping condition is simply, $a_1 = a_2 = R_1\alpha_1 = R_2\alpha_2$, since the ropes unwind from the pulleys without sliding, and the two cog wheels don't slip between them. For the signs to make sense, we have defined some axes for the accelerations and the turning, so that they are all positive when both blocks move downwards. That's the simplest.

b) What is the acceleration of the blocks? What is the angular acceleration of each cog wheel/pulley?

Solution: Write down the force/torque equation for each subsystem:

$$m_1 a = m_1 g - T_1, \tag{13}$$
$$m_2 a = m_2 g - T_2, \tag{14}$$
$$I_1\alpha_1 = R_1 T_1 - f R_1, \tag{15}$$
$$I_2\alpha_2 = R_2 T_2 + f R_2. \tag{16}$$

Solving for a, using the constraints and the moments of inertia of the discs, we find

$$a = \frac{m_1 + m_2}{m_1 + m_2 + \frac{1}{2}M_1 + \frac{1}{2}M_2} g = \frac{2}{3} g = 6.53 \text{ m/s}^2. \tag{17}$$

That gives $\alpha_1 = 16.3 \text{ s}^{-2}$ and $\alpha_2 = 21.8 \text{ s}^{-2}$. One could get the same from energy conservation.

We now add a third cog wheel between the first two, of mass $M_3 = 1.00$ kg and radius $R_3 = 0.350$ m. It is also a uniform disc. See Figure (right).

c) What is now the acceleration of the blocks?

Solution: Now block 2 goes up, and block 1 goes down. If measuring a_1 positive down, a_2 positive up, α_1 and α_2 positive counter-clockwise, and α_3 positive clockwise, we have $a_1 = a_2 = R_1\alpha_1 = R_2\alpha_1 = R_3\alpha_3$. Setting up again the equations, but now with two contact forces between the pulleys, and keeping in mind Newton 3., we get

$$m_1 a = m_1 g - T_1, \tag{18}$$
$$m_2 a = T_2 - m_2 g, \tag{19}$$
$$I_1\alpha_1 = R_1 T_1 - f_1 R_1, \tag{20}$$
$$I_2\alpha_2 = f_2 T_2 - T_2 R_2, \tag{21}$$
$$I_3\alpha_3 = R_3(f_1 - f_2). \tag{22}$$

Solving for a with the constraints and the moments of inertia, we find

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}(M_1 + M_2 + M_3)} g = \frac{1}{11} g = 0.891 \text{ m/s}^2. \tag{23}$$

Again, energy conservation is perhaps an even simpler way to get the same.

d) Now we imagine that the third wheel is turned by an engine, providing a torque of 10.0 Nm, counterclockwise. What is now the acceleration of the blocks?

Solution: We do exactly the same as before but now the last equation is different

$$m_1 a = m_1 g - T_1, \tag{24}$$
$$m_2 a = T_2 - m_2 g, \tag{25}$$
$$I_1\alpha_1 = R_1 T_1 - f_1 R_1, \tag{26}$$
$$I_2\alpha_2 = f_2 T_2 - T_2 R_2, \tag{27}$$
$$I_3\alpha_3 = R_3(f_1 - f_2) - \tau. \tag{28}$$

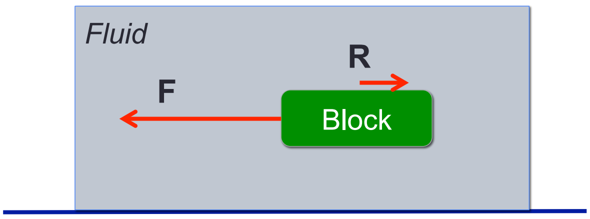
Adding up, we find

$$a = \frac{(m_1 - m_2)g - \tau/R_3}{m_1 + m_2 + \frac{1}{2}(M_1 + M_2 + M_3)} = \frac{1}{11}g - \frac{\tau}{11R_3} = -2.14 \text{ m/s}^2. \qquad (29)$$

For each question, provide a sketch, an algebraic expression, as well as the numerical result.

Typical mistakes: Half of you missed the relation $a_1 = a_2$. Some had instead $\alpha_1 = \alpha_2$, which gives a result, but the wrong one. Some solved for the two wheels separately, using relations like $T_1 = T_2$ (which is wrong). Partial credit for any calculation that gets to a final number, or expressions for a_1, a_2, however wrong. Many had extra R_1 and R_2 in the final results, that shouldn't be there, or ignored the masses and moments of inertia of the wheels. Some had simple sign errors. A few thought that the pulleys themselves were falling under gravity(!) Quite a few tried to solve it using the torques around the centers of the wheels. Which is clever, but things got mixed up in the process. Credit for that. Quite a few ended up with accelerations larger than g. Some of you commented on that being unreasonable. Credit for that. 34% of the total available points were awarded overall.

Problem 4: What a Drag! (22 percent)



A block of mass $M = 1.00$ kg is being dragged through some viscous fluid by an external force $F = 10.0$ N. The resistive force can be written as $R = -bv$, where v is the speed and $b = 4.00$ kg/s is a phenomenological constant. You may ignore gravity (we imagine that the block is floating inside the fluid). The following integrals may come in handy:

$$\int \frac{dz}{z} = \ln(z), \qquad \int dt\, e^{-at} = -\frac{1}{a}e^{-at}. \qquad (30)$$

a) Using the three physical quantities provided, use dimensional analysis to find the terminal velocity of the motion v_T and the characteristic time τ.

Solution: By comparing units, one finds for a speed and a time:

$$v_T = \frac{F}{b} = 2.50 \text{ m/s}, \qquad \tau = \frac{M}{b} = 0.250 \text{ s}. \tag{31}$$

b) Assuming that the block starts from rest, find $v(t)$.

Solution: Newton 2. becomes

$$M\frac{dv}{dt} = F - bv. \tag{32}$$

This differential equation has the solution (use any procedure you like to find it)

$$v(t) = v_T \left(1 - e^{-t/\tau}\right), \qquad v_T = \frac{F}{b}, \qquad \tau = \frac{M}{b}. \tag{33}$$

c) Find the distance travelled as a function of time $x(t)$.

Solution: Perform the integral

$$x(t) = \int dt v(t) = v_T\tau \left(\frac{t}{\tau} + e^{-t/\tau}\right). \tag{34}$$

Some of the work done by the force F on the system becomes internal energy, some becomes kinetic energy.

d) How much internal energy has been generated by the time the block reaches half the terminal velocity, $v(t) = v_T/2$? How big a fraction of the total work is that?

Solution: The work done is

$$W = Fx(t) = K + E_{\text{int}} = \frac{1}{2}Mv^2(t) + E_{\text{int}}. \tag{35}$$

So we need to find $x(t)$ and $v(t)$ evaluated at the time when $v(t) = v_T/2$. So we write

$$\frac{1}{2}v_T = v_T\left(1 - e^{-t/\tau}\right) \to e^{-t/\tau} = \frac{1}{2} \to t/\tau = \ln(2). \tag{36}$$

Inserting this, we find

$$E_{\text{int}} = Fv_T\tau\left(\ln(2) + \frac{1}{2}\right) - \frac{1}{2}M\left(\frac{v_T}{2}\right)^2 = Fv_T\tau\left(\ln(2) + \frac{1}{2} - \frac{1}{8}\right) = 6.68 \text{ J}. \tag{37}$$

The fraction is then

$$\frac{E_{\text{int}}}{W} = \frac{\ln(2) + \frac{1}{2} - \frac{1}{8}}{\ln(2) + \frac{1}{2}} = 0.895. \tag{38}$$

For each question, provide a sketch, an algebraic expression, as well as the numerical result.

Typical mistakes: Many of you never got this far. Many of the rest managed to figure out the terminal velocity, very few the characteristic time (dimensional analysis was almost not used at all). The vast majority gave up on solving for $v(t)$. A few took a chance at using 1-D kinematics with constant acceleration, but that was wrong. Some mentioned that they would integrate $v(t)$ to find $x(t)$ giving at least partial credit. And some had an idea about the principle of d), but no-one got the right result. 12% of the total available points were awarded overall. Perhaps worth reading up on this chapter/my notes!