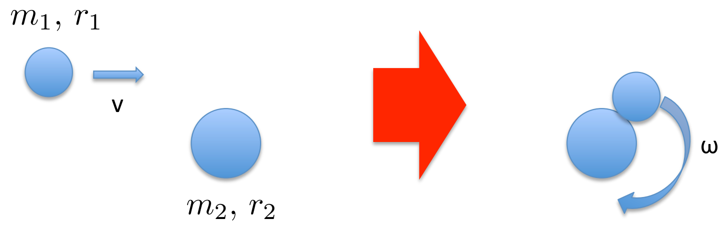
Høsten 2018

FYS100 Fysikk
Eksamen/Exam
Solutions

Problem 1: A puck and another puck...(25 percent)



A puck of mass $m_1 = 80.0$ g and radius $r_1 = 4.00$ cm glides across an air table at a speed $\mathbf{v} = 1.50$ m/s as shown in the Figure. It makes a glancing collision with a second puck of radius $r_2 = 6.00$ cm and mass $m_2 = 120$ g (initially at rest) such that their rims just touch. Because their rims are coated with instant-acting glue the pucks stick together and rotate after the collision.

a) What is the angular momentum of the system relative to the center of mass?

Solution: First, find the center of mass. Putting a coordinate system at the center of m_2, the particle m_1 has the time-dependent position vector $(x(t), r_1 + r_2)$. The CM is then at (with silly notation, where r_1 and r_2 are the radii of the discs, but $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of those two discs)

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}(x(t), r_1 + r_2). \tag{1}$$

The angular momentum of m_1 around the CM is then $(\mathbf{r}_1 - \mathbf{r}_{CM}) \times \mathbf{p}_1$, which makes $x(t)$ vanish, and $y = r_1 + r_2 - y_{CM}$, so that

$$|L| = m_1 v \left[r_1 + r_2 - \frac{m_1}{m_1 + m_2}(r_1 + r_2) \right] = \frac{m_1 m_2}{m_1 + m_2}(r_1 + r_2)v = 7.2 \times 10^{-3} \text{ kg m}^2/\text{s}. \tag{2}$$

b) What is the angular speed about the center of mass?

Solution: After the collision, we have $|L| = I\omega$, with I the complete moment of inertia around the CM. Using the parallelaxis theorem, this is

$$I = \frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2 + m_1 D_1^2 + m_2 D_2^2, \tag{3}$$

where D_1 and D_2 are distances from the center of each disc to the CM, so that

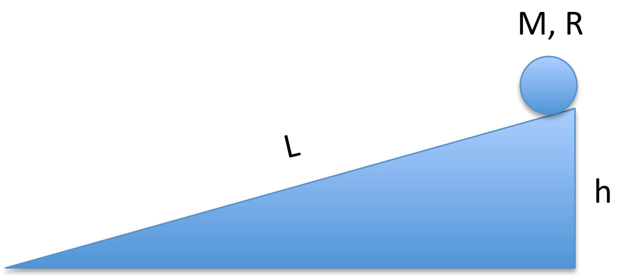
$$I = \frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2 + m_1(y_1 - y_{CM})^2 + m_2(y_2 - y_{CM})^2 =$$
$$\frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2 + \frac{m_1 m_2}{m_1 + m_2}(r_1 + r_2)^2 = 7.6 \times 10^{-4} \text{ kg m}^2. \tag{4}$$

Then

$$\omega = \frac{|L|}{I} = 9.47 \text{ s}^{-1}. \tag{5}$$

Typical mistakes: Not knowing $r \times p$ but using $I\omega$ throughout. Forgetting to compute it relative to the CM. Using energy conservation (might work out, but then need the linear motion as well). Computational mistakes. Based on chapters 9, 10 and 11. Hand-in problem, from book.

Problem 2: Rolling down a hill. (25 percent)



A hollow cylinder, a solid cylinder and a sphere roll down a hill. They all have mass $M = 10.0$ kg and radius $R = 1.00$ m, and start at rest a distance $L = 10.0$ m up the hill, corresponding to a difference in vertical height between top and bottom of $h = 1.00$ m. The moment of inertia of a hollow cylinder is MR^2, of a solid cylinder $MR^2/2$, and of a sphere $2/5 \times MR^2$. There is gravity $g = 9.80$ m/s^2.

a) What is the speed of each of the objects at the bottom of the hill? How long does it take each of them to get to the bottom?

Solution: Straight-forward use of energy conservation: potential energy is converted to kinetic energy. Remembering the rotational energy component gives:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2\left(1 + \frac{I}{MR^2}\right), \tag{6}$$

where we have used the rolling without sliding condition $v = R\omega$. We then have

$$v = \sqrt{\frac{2gh}{1 + \frac{I}{MR^2}}} = \sqrt{\frac{2gh}{(2, 3/2, 7/5)}}, \tag{7}$$

inserting the moments of inertia for the three objects. This gives 3.13 m/s, 3.61 m/s, 3.74 m/s.
To find the time use 1-D kinematics to get to the distance L,

$$t = \frac{L}{v_{\text{avg}}} = \frac{L}{(v-0)/2} = \frac{2L}{v}, \tag{8}$$

giving 6.39 s, 5.53 s, 5.35 s.

b) What would happen to the results in a) if all the masses M and radii R were no longer the same? (You may ignore the small gravitational potential difference this causes at the initial point, (difference in R)$\ll h$).

Solution: Nothing. M and R drop out of the equations.

c) What is the minimum static friction coefficient μ_s required for each of them to continue to roll without sliding? Does this depend on M and/or R?

Solution: There is a force of friction f_s up the hill providing torque and acceleration. There is a component of gravity down the hill $Mg\sin\theta$. And a component of gravity into the incline, cancelled by the normal force, so that.We can then write down the equations

$$\begin{aligned} Mg\sin\theta - f_s &= Ma, && \text{Linear acceleration,} && (9)\\ Rf_s &= I\alpha, && \text{Angular acceleration,} && (10)\\ R\alpha &= a, && \text{Rolling without sliding,} && (11)\\ |n| &= Mg\cos\theta, && \text{Normal force,} && (12)\\ f_s &= \mu_s|n|, && \text{Largest static friction.} && (13) \end{aligned}$$

This gives

$$\mu_s = \frac{\tan\theta}{1 + \frac{MR^2}{I}}. \tag{14}$$

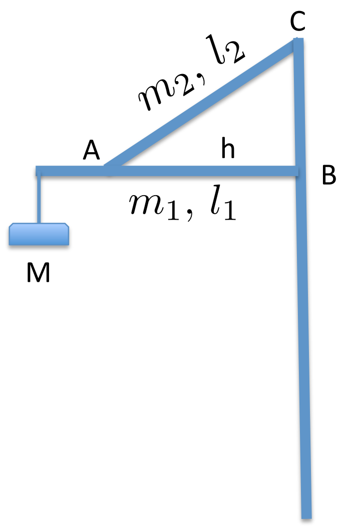
Once again, M and R cancel out. Inserting the three values of I/MR^2 and using the geometry of the triangle to find

$$\tan\theta = \frac{1}{\sqrt{10^2 - 1^2}} \tag{15}$$

we get $\mu_s = 0.0503; 0.0335; 0.0287$ respectively.

Typical mistakes: Not using energy conservation but forces (can work, but has to be done correctly). Forgetting rotational energy omponent. Forgetting linear energy component. Forgetting no-sliding condition. Thinking that there is rolling along a horizontal surface (then there is no acceleration and so no force or friction). Computational mistakes. Based on chapters 2, 4, 5, 6, 7, 8 and 10.

Problem 3: Street light in equilibrium (25 percent)

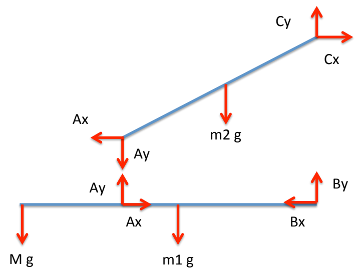


A lamp post consists of a vertical pole, a horizontal bar of mass m_1 and length l_1 attached to it at a point B, and a diagonal bar of mass m_2 and length l_2 attached to it at a point C, as shown in the Figure. The horizontal bar is attached to the diagonal one at a point A, a distance h from point B. At the end of the horizontal bar, a lamp of mass M has been attached, hanging straight down. The system is in static equilibrium.

a) Decomposing the forces at points A, B and C along the vertical and horizontal, write down the relevant force equations for the horizontal bar and the diagonal bar to stay at rest.

Solution: There are two systems: The horizontal and the diagonal bars. See Figure for choice of direction of forces (there are other possibilities, but not too many, if along the horizontal and vertical). Keep in mind that the A-force is opposite for the two systems (Newton 3.). We can then write down:

$$
\begin{aligned}
F_A^x &= F_B^x, \quad \text{no net force in x for horizontal bar} && (16)\\
F_A^x &= F_C^x, \quad \text{no net force in x for diagonal bar} && (17)\\
F_B^y + F_A^y &= Mg + m_1 g, \quad \text{no net force in y for horizontal bar} && (18)\\
F_C^y - F_A^y &= m_2 g. \quad \text{no net force in y for diagonal bar} && (19)
\end{aligned}
$$



b) Write down the torque equations, so that both horizontal bar and diagonal bar do not rotate. For each of the two bars, use an axis at A.

Solution: Torques of force A around A is zero. For the horizontal bar, F_B^x also gives zero torque, because it is parallel to the arm. That leaves:

$$
\begin{aligned}
\tau_{m_1} &= -m_1 g \left(h - \frac{l_1}{2} \right), && (20)\\
\tau_M &= Mg(l_1 - h), && (21)\\
\tau_{By} &= F_B^y h, && (22)
\end{aligned}
$$

which must sum to zero.

$$
F_B^y h + Mg(l_1 - h) - m_1 g \left(h - \frac{l_1}{2} \right) = 0 \qquad (23)
$$

For the diagonal bar, we have

$$
\begin{aligned}
\tau_{m_2} &= -m_2 g \frac{l_2}{2} \cos\theta, && (24)\\
\tau_{Cx} &= -F_C^x l_2 \sin\theta, && (25)\\
\tau_{Cy} &= F_C^y l_2 \cos\theta, && (26)
\end{aligned}
$$

which must also sum to zero

$$F_C^y l_2 \cos\theta - F_C^x l_2 \sin\theta - m_2 g \frac{l_2}{2} \cos\theta = 0 \tag{27}$$

c) Solve the equations to find the forces at A, B, and C.

Solution: You solve all six equations, and get

$$F_A^y = \left[M\frac{l_1}{h} + m_1\frac{l_1}{2h}\right] g, \tag{28}$$

$$F_B^y = \left[M\left(1 - \frac{l_1}{h}\right) + m_1\left(1 - \frac{l_1}{2h}\right)\right] g, \tag{29}$$

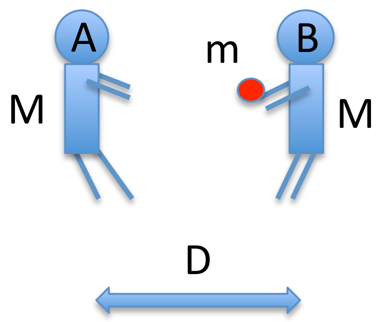
$$F_C^y = \left(M\frac{l_1}{h} + m_1\frac{l_1}{2h} + m_2\right) g, \tag{30}$$

$$F_A^x = F_B^x = F_C^x = \left(M\frac{l_1}{h} + m_1\frac{l_1}{2h} + \frac{m_2}{2}\right)\frac{g}{\tan\theta}, \tag{31}$$

$$\tag{32}$$

Typical mistakes: Not realising that there are two systems. Not decomposing along horizontal and vertical (could work, but has to be done correctly). Not decomposing along orthogonal directions. Forgetting gravity. Computational mistakes. Doing three systems. Not realising that the A-force acts on both systems with opposite direction.

Problem 4: Ball games in space, Part 1. (25 percent)



Two astronauts A and B, each of mass M, find themselves in space, a distance D apart, and at rest relative to each other. B is holding a ball of mass m.

a) Where is the center of mass (CM) of the astronaut-astronaut-ball system, in a coordinate system with origin at the position of A? We will take the coordinate x to increase in the direction from A to B.

Solution: CM is at

$$x_{CM} = \frac{m_A r_A + m_B r_B + m_{ball} r_{ball}}{m_A + m_B + m_{ball}} = \frac{M+m}{2M+m} D. \tag{33}$$

B now throws the ball towards A, with a speed v.

b) What is the velocity of B after throwing the ball? Where is the CM of the entire system?

Solution: Conservation of momentum of the ball-B system gives

$$MU_B + mv = 0 \rightarrow U_B = -\frac{mv}{M}. \tag{34}$$

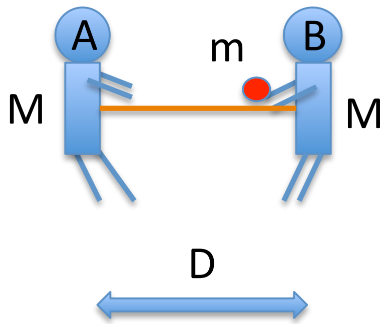
CM of whole A-B-ball system is unchanged, because it is an isolated system. Objects inside may move around, but the overall CM stays put.

c) A catches the ball. What is then the velocity of A + ball? Where is the CM of the entire system?

Solution: Conservation of momentum in A-ball system gives

$$(M+m)U_A = mv \rightarrow U_A = \frac{mv}{M+m}. \tag{35}$$

CM of whole system is still unchanged.



Now consider the same initial situation, but imagine that the astronauts had been connected by a massless, unstretchable string.

d) What is the velocity of B after he has thrown the ball? Where is then the CM of the whole system? What is the velocity of A after he catches the ball?

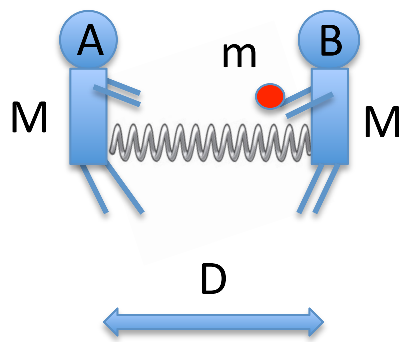
Solution: Now it is the A+B system, that effectively throws the ball, so it is like b) but with $2M$ instead of M.

$$2MU_B + mv = 0 \to U_B = -\frac{mv}{2M}. \tag{36}$$

CM of whole system is unchanged, and so after he catches the ball, and all relative motion has stopped, all speeds must be zero, $U_A = 0$, as for the initial state.

Typical mistakes: Using energy conservation instead of momentum. Not realising that CM stays put always, and calculating with time-dependent positions of ball and A and B...it could work but then has to be done correctly. Thinking that there is gravity and it is projectile motion. Based on chapter 9.

Problem 5: Ball games in space, Part 2 (25 percent)



Consider the pair of astronauts in problem 4 d), but imagine that the connection was not an unstretchable string, but a massless spring with spring constant k, where we will assume that the equilibrium length is D, the initial separation. In the following, neglect the time it takes the ball to leave the hand, relative to the time it takes the spring force to react.

a) Taking the result of 4 b) for the velocity of B (if you didn't solve it, use the symbol U_B), go to the center of mass (CM) frame of the astronaut-astronaut-only system (not including the ball!). Find the position and velocity of the ball in that frame as a function of time (up until it is caught by A).

Solution: In the original frame with B moving with U_B and U_A being stationary, the CM of the A-B system moves with $U_B/2$. Going to the CM frame, the initial velocities are

$$v' = v - U_B/2 = v\left(1 + \frac{m}{2M}\right), \tag{37}$$

$$U'_A = 0 - U_B/2 = \frac{mv}{2M}, \tag{38}$$

$$U'_B = U_B - U_B/2 = -\frac{mv}{2M}. \tag{39}$$

This makes sense, since A and B have the same mass, and move symmetrically to the left and the right. The ball does not accelerate until it gets caught, and so its position is given by the constant speed equation, starting at position $D/2$

$$x_v(t) = D/2 + vt\left(1 + \frac{m}{2M}\right). \tag{40}$$

b) Still in the CM frame, show that the separation between A and B can be described as a harmonic oscillator. Find the angular frequency, ω. What are the initial conditions, corresponding to just after the ball has left the hand of B?

Solution: The stretching of the spring is in terms of the combination (neglecting ' for CM frame)

$$\Delta x = x_B - x_A - D. \tag{41}$$

Note also that $x_A = -x_B$. Stoke's law, and the orientation of the axes give for the two masses

$$Ma_A = k\Delta x, \tag{42}$$

$$Ma_B = -k\Delta x. \tag{43}$$

This gives

$$\frac{d^2\Delta x}{dt^2} = -\frac{2k}{M}\Delta x, \tag{44}$$

which is a harmonic oscillator with $\omega = \sqrt{2kM}$. The initial condition is that $\Delta x(0) = 0$ and $d\Delta x/dt(0) = -mv/M$.

c) Find the position of A in the CM-frame as a function of time. Write down an equation for the time t_{catch} when the ball reaches the position of A. You do not need to solve the equation.

Solution: The harmonic oscillator solution is

$$\Delta x = a \cos \omega t + b \sin \omega t, \qquad a = 0, \quad b = \frac{-mv}{M\omega}. \tag{45}$$

Also, $x_A = -D/2 - \Delta x/2$. Then the catch happens (the first time) when

$$x_A = x_v \to -\frac{D}{2} + \frac{mv}{2M\omega} \sin \omega t = \frac{D}{2} + vt \left(1 + \frac{m}{2M}\right). \tag{46}$$

Typical mistakes: Very few solved this problem. Some of you wrote down the harmonic oscillator solution, and an expression for ß*omega*. But very few managed to go to the CM frame and realised what the correct frequency was. This problem is extra points in addition to the 100% from problem 1-4.