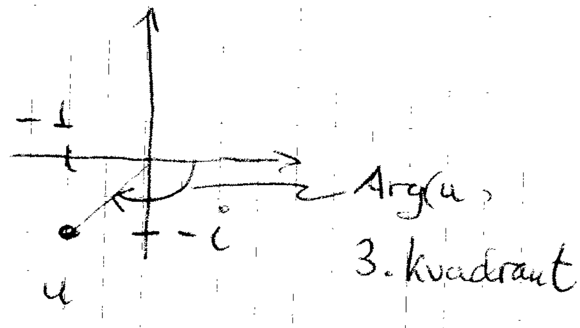
AMA 100 – H2008

1

a) $u = -1 - i$



$$r = |u| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arg(u) = \tan^{-1}\underbrace{\left(\frac{-1}{-1}\right)}_{1} - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

$\Rightarrow$

$$-1 - i = \underline{\underline{\sqrt{2}\, e^{-i3\pi/4}}}$$

(Her har vi brukt hovedargumentet $-3\pi/4$, <u>kan også bruke</u> f. ex $\arg(u) = \frac{5\pi}{4}$.)

b) $(1+i)z = i$

$$z = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i - i^2}{1^2 + 1^2} = \frac{1+i}{2} = \underline{\underline{\frac{1}{2} + \frac{1}{2}i}}$$

[<u>Prøve</u>: $(1+i)\frac{(1+i)}{2} = \frac{1^2 + 2i + i^2}{2} = \frac{2i}{2} = i$ Ok]

c) $$\frac{z_1^2}{z_2} = \frac{\left(2e^{i2\pi/3}\right)^2}{4e^{i\pi/2}} = \frac{\cancel{4}\, e^{i4\pi/3}}{\cancel{4}\, e^{i\pi/2}} \quad \text{(de Moivre)}$$

$$= e^{i(4\pi/3 - \pi/2)} = e^{i5\pi/6}$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \underline{\underline{-\frac{\sqrt{3}}{2} + \frac{1}{2}i}}$$

d) $$w^3 = e^{i3\pi/4}. \qquad n = 3, \quad r = 1, \quad \theta = \frac{3\pi}{4}, \quad \theta/3 = \frac{\pi}{4}$$

Formelark $\Rightarrow$ $$w = e^{i\left(\frac{\pi}{4} + k\cdot\frac{2\pi}{3}\right)}, \quad k = 0, 1, 2$$

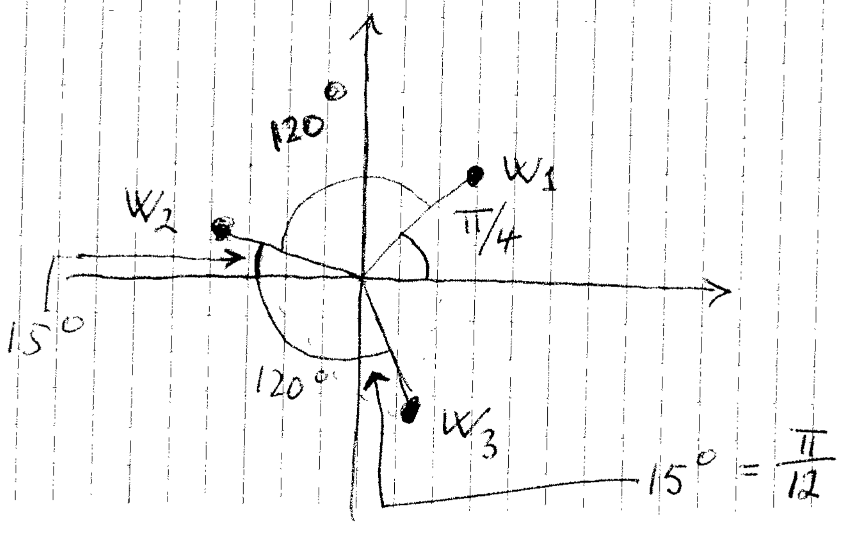
$k = 0$ $$w_1 = \underline{\underline{e^{i\pi/4}}}$$

$k = 1$ $$w_2 = e^{i\left(\pi/4 + \frac{2\pi}{3}\right)} = \underline{\underline{e^{i11\pi/12}}}$$

$[11\pi/12 = 165^\circ]$

$k = 2$ $$w_3 = e^{i\left(\pi/4 + 2\cdot\frac{2\pi}{3}\right)} = \underline{\underline{e^{i19\pi/12}}}$$

$[19\pi/12 = 285^\circ]$



(2)

a) $y'' - 2y' + 10y = 0$

$r^2 - 2r + 10 = 0$

$\underbrace{r^2 - 2r + 1}_{(r-1)^2} - 1 + 10 = 0$

$(r-1)^2 = -9$

$a = 1$ $\quad$ $b = 3$

$r - 1 = \pm 3i, \quad \underline{r = 1 \pm 3i}$ (Case III)

$\Rightarrow \quad \underline{\underline{y = Ce^x \cos(3x) + De^x \sin(3x)}}$

b) $y'' - 3y' + 2y = 4x, \quad y(0) = y'(0) = 0$

$r^2 - 3r + 2 = 0 \Rightarrow r_1 = 1, r_2 = 2$ (Case I)

$\Rightarrow \quad \underline{y_h = Ce^x + De^{2x}}$

$f(x) = 4x$ = polynom av grad 1 (ingen exponential!)
dvs $\underline{a=0}$

$\Rightarrow \quad \left.\begin{aligned} y_s &= Ax + B \\ y_s' &= A, \quad y_s'' = 0 \end{aligned}\right\}$ sätt in:

$\left.\begin{aligned} 0 - 3 \cdot A + 2(Ax + B) &= 4x \\ 2Ax + 2B - 3A &= 4x \end{aligned}\right\} \quad \begin{aligned} 2A &= 4 \\ 2B - 3A &= 0 \end{aligned}$

dvs $\boxed{A = 2}, \boxed{B = 3}$

$\underline{y_s = 2x + 3}$

$$\Rightarrow \quad y = y_h + y_s = Ce^{x} + De^{2x} + 2x + 3$$

//

$$y(0) = Ce^{0} + De^{0} + 2\cdot 0 + 3 = 0$$ gitt

$$C + D = -3$$

$$y' = Ce^{x} + 2De^{2x} + 2$$

$$y'(0) = C\cdot e^{0} + 2De^{0} + 2 = 0$$ gitt

$$C + 2D = -2$$

$$\Rightarrow \boxed{D = 1}, \quad \boxed{C = -4}$$

Entydig løsn av IVP

$$y = -4e^{x} + e^{2x} + 2x + 3$$

c)

$$y' = 2(y-20), \quad y(0) = 50$$

$$\frac{dy}{dx} = 2(y-20)$$

$$\frac{dy}{y-20} = 2, \quad \int \frac{dy}{y-20} = \int 2\,dx$$

$$\ln|y-20| = 2x + C$$

$x=0,\ y=50$: $\ln|50-20| = 2\cdot 0 + C \Rightarrow C = \ln 30$

$$\Rightarrow \quad y - 20 = e^{2x + \ln 30} = 30e^{2x}$$

$$y = 30e^{2x} + 20$$

(3)

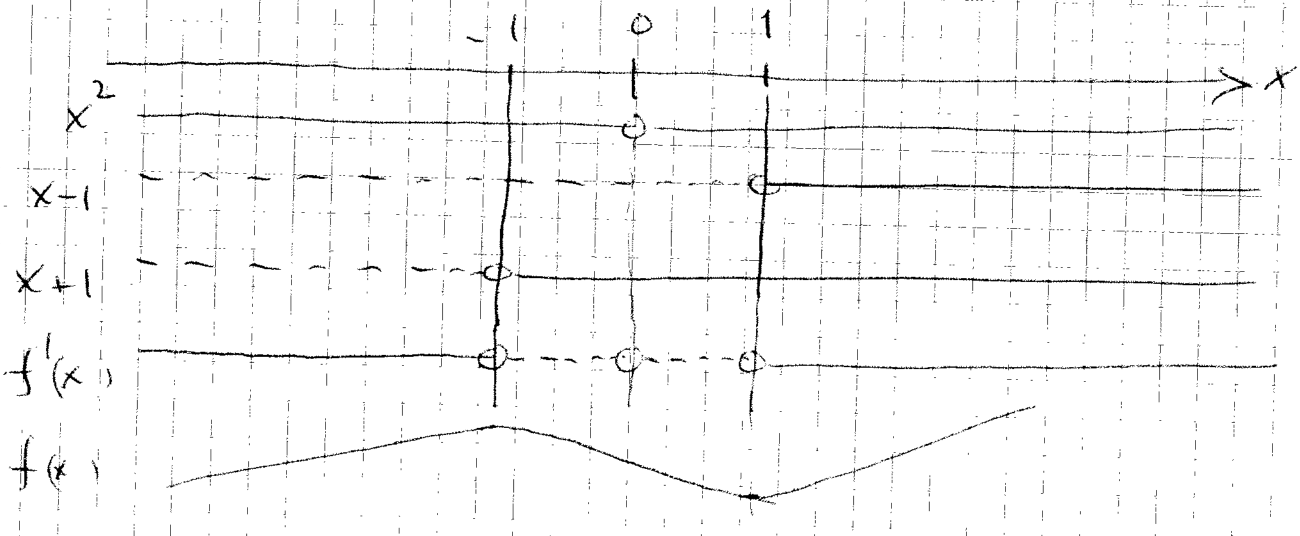
a) $\lim\limits_{x \to 0} \dfrac{-e + e^{-x}}{2x} \underset{\left[\frac{0}{0}\right]}{\overset{\text{l'Hop}}{=}} \lim\limits_{x \to 0} \dfrac{-e^{x} - e^{-x}}{2} = \dfrac{-2}{2} = \underline{\underline{-1}}$

b) $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{\sqrt{x}} \underset{\left[\frac{\infty}{\infty}\right]}{=} \lim\limits_{x \to \infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x \to \infty} \dfrac{2}{\sqrt{x}} = \underline{\underline{0}}$

(4)

a) $f(x) = 3x^5 - 5x^3$, $D = \mathbb{R}$

$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x-1)(x+1)$



Ser: $x = -1$ lok maxpkt. } er de eneste
$x = 1$ lok minpkt. } lokal ekstremalpunkter.

$f(x) = 3x^5 - 5x^3$ = polynom av grad 3

$\underline{3 = \text{odde tall}} \Rightarrow$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$

dvs $f(x)$ kan få så liten og så stor funksjonsverdi vil vil $\Rightarrow$ ingen global ekstremalpkt (når vi antar $D = \mathbb{R}$)

b) (*) $x^2 + (y-1)^2 = 5^2$

$P(-3,-3)$:

$$(-3)^2 + (-3-1)^2 = 3^2 + 4^2 = 9+16 = 25 = \underline{5^2}$$

Ok!

$\frac{dy}{dx} = ?$

Ta $\frac{d}{dx}$ av (*): $\cancel{2}x + \cancel{2}(y-1)\cdot y' = 0$

$$(y-1)y' = -x$$

$$\frac{dy}{dx} = y' = \frac{-x}{y-1}$$

$\Rightarrow$ Stign. for tangenten i P: $\frac{-(-3)}{-3-1} = -\frac{3}{4}$

$\Rightarrow$ Lign for tangenten

$$y - (-3) = \left(-\frac{3}{4}\right)(x-(-3))$$

$$\underline{y = -\frac{3}{4}x - \frac{21}{4}}$$

c) $x^3 + \cos(xy) - y = 2x$

Ta $\frac{d}{dx}$: $3x^2 - \sin(xy) \cdot [1 \cdot y + x \cdot y'] - y' = 2$

$y' = \frac{dy}{dx}$; multipliser ut:

$$3x^2 - y\sin(xy) - y'x\sin(xy) - y' = 2$$

$$y'(-x\sin(xy) - 1) = 2 - 3x^2 + y\sin(xy)$$

$$y' = \frac{2 - 3x^2 + y\sin(xy)}{-x\sin(xy) - 1}$$

eller $$\frac{dy}{dx} = y' = \underline{\underline{\frac{3x^2 - 2 - y\sin(xy)}{x\sin(xy) + 1}}}$$

5

a) $$\int xe^x\,dx = xe^x - \int 1 \cdot e^x\,dx$$ [delvis integrasjon]

$(x)'$

e^x

$$= xe^x - e^x + C = \underline{\underline{(x-1)e^x + C}}$$

[Prøve: $((x-1)e^x + C)' = 1 \cdot e^x + (x-1)e^x + 0 =$

$= e^x + xe^x - e^x = \underline{xe^x}$ $\underline{\text{ok}}$!]

b) $\int \frac{x}{\sqrt{x^2-1}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\,\frac{du}{2\cancel{x}}$

Substitusjon:

$\boxed{u = x^2 - 1}$

$\frac{du}{dx} = 2x$

$\boxed{dx = \frac{du}{2x}}$

$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$

$= u^{\frac{1}{2}} + C = \underline{\underline{\sqrt{x^2-1} + C}}$

c) $\int \frac{x^2+1}{x^2(x+1)}\,dx = ?$ delbrøk

$\frac{x^2+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

F. eks Felles brøkstrek

$x^2+1 = Ax(x+1) + B(x+1) + Cx^2$

$x^2+1 = (A+C)x^2 + (A+B)x + B$

$\Rightarrow A+C = 1, \quad A+B = 0, \quad \boxed{B = 1} \Rightarrow$

$\boxed{A} = -B = \boxed{-1}$

$\boxed{C} = 1-A = \boxed{2}$

$A=-1 \quad B=1 \quad C=2$ (9)

$$\Rightarrow \int \frac{x^2+1}{x^2(x+1)}\,dx = \int -\frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{2}{x+1}\,dx$$

$$= -\ln|x| - \frac{1}{x} + 2\ln|x+1| + C$$

[Kan sjekke B og C ved innsettingsmetoden, men ikke A.]

d) $$\int \frac{x}{\underbrace{x^2-2x+5}}\,dx$$

Komplekse røtter!

$$x^2-2x+5 = 0$$

$$x^2-2x+1-1+5 = 0$$

$$\underbrace{\{(x-1)^2+4\}}_{\text{fullstendig kvadrat!}} = 0 \quad \left[\Rightarrow x-1 = \pm 2i \quad x = 1 \pm 2i\right]$$

fullstendig kvadrat!

$$= \int \frac{x}{(x-1)^2+4}\,dx = \int \frac{u+1}{u^2+4}\,du$$

$\boxed{u = x-1} \Rightarrow \boxed{x = u+1}$

$\boxed{du = dx}$

$(a=2)$

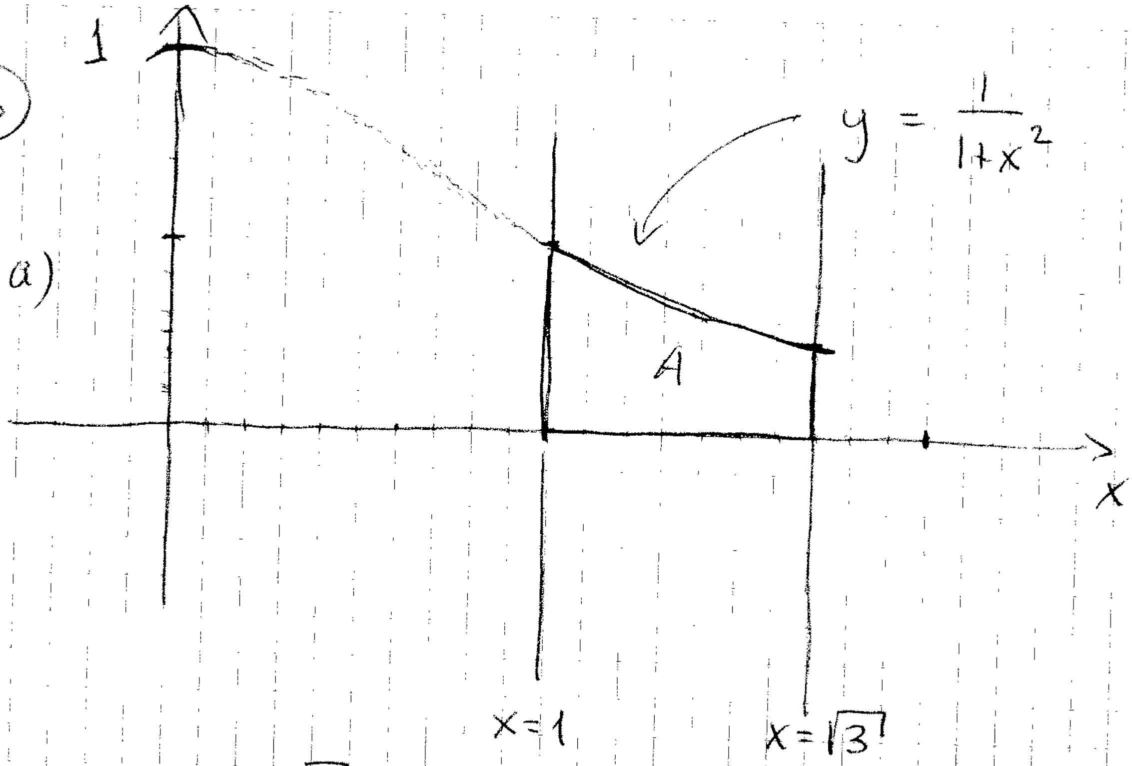
$$= \int \frac{u}{u^2+\underset{2^2}{4}}\,du + \int \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\ln(u^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\ln(x^2-2x+5) + \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

(6)

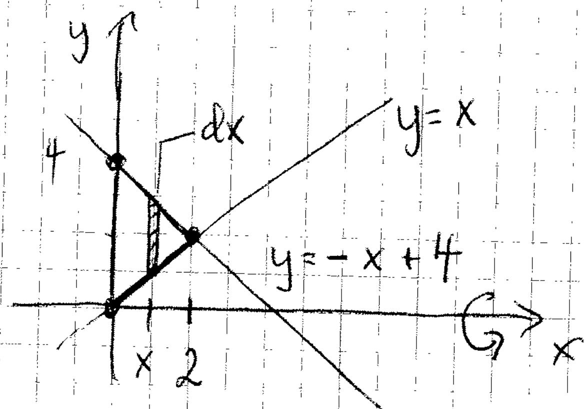
a)



$$A = \int_1^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_1^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1) = \frac{\pi}{3} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{12}}}$$

b)



$$\text{volum} = \int_0^2 \pi(R^2 - r^2)\,dx = \int_0^2 \pi\left((-x+4)^2 - x^2\right)dx$$

$$= \pi\int_0^2 8(2-x)\,dx = 8\pi\left(2x - \tfrac{1}{2}x^2\right)\Big|_0^2$$

$$= 8\pi\left(2\cdot 2 - \tfrac{1}{2}\cdot 2^2 - 0\right) = \underline{\underline{16\pi}}$$

7 a) 1) $A = \pi r^2$

2) $C = 2\pi r$

oppgitt : $\frac{dA}{dt} = 4 \; (\text{cm}^2/\text{min})$

$r = 2 \; (\text{cm})$

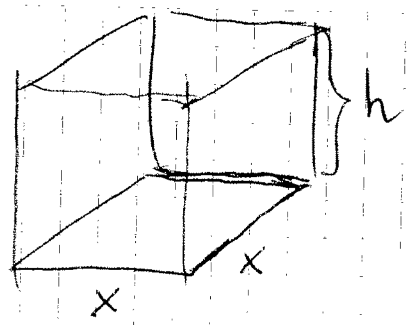
Finne : $\frac{dC}{dt} = ?$

1') $\frac{d}{dt}$ av 1) $\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2') $\frac{d}{dt}$ av 2) $\Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$

1') $\underset{\substack{\uparrow \\ 4}}{\frac{dA}{dt}} = 2\pi \underset{\substack{\uparrow \\ 2}}{r} \frac{dr}{dt} \Rightarrow \underline{\frac{dr}{dt}} = \frac{4}{2\pi 2} = \underline{\frac{1}{\pi}}$

2') $\frac{dC}{dt} = 2\pi \cdot \underset{\substack{\uparrow \\ \frac{1}{\pi}}}{\frac{dr}{dt}} = 2\pi \cdot \frac{1}{\pi} = \underline{\underline{2 \text{ cm}/\text{min}}}$

b) 

1) $V = x^2 h = 8$

2) A = total overflate $= 2x^2 + 4xh$

A minimum = ?

1) $\Rightarrow \quad h = \dfrac{8}{x^2}$, sett inn i 2) $\Rightarrow$

$$\boxed{A} = 2x^2 + 4x \cdot \frac{8}{x^2} = \boxed{2x^2 + \frac{32}{x}}, \quad \underline{x > 0.}$$

A min. $\Rightarrow$

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{4(x^3 - 8)}{x^2} \equiv 0$$

$$\Rightarrow x^3 - 8 = 0, \text{ dvs } \underline{x = 2}$$

Kun ett kritisk pkt $\Rightarrow$ A må ha min. verdi for $x = 2$ (for $\lim\limits_{x \to 0^+} A = \infty$, $\lim\limits_{x \to \infty} A = \infty$)

minverdi $= A(2) = 2 \cdot 2^2 + \dfrac{32}{2} = \underline{\underline{24}}$