Eksamen i

Matematiske Metoder 1, UiS

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Løsningsforslag

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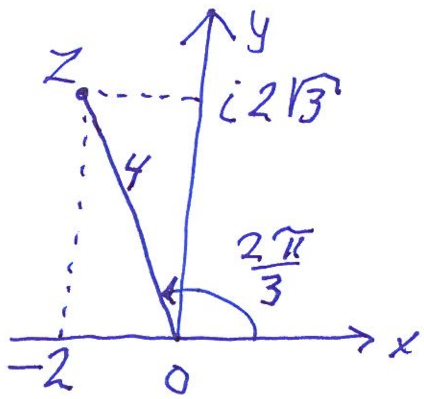
Oppgave 1

a) $\underline{\underline{v}} = (2-3i)\cdot(4+4i) = 8+8i-12i+12 = \underline{\underline{20-4i}}$

$$\underline{\underline{w}} = \frac{1+2i}{1+i} = \frac{(1+2i)(1-i)}{(1+i)\cdot(1-i)} = \frac{3+i}{1+1} = \underline{\underline{\frac{3}{2}+\frac{1}{2}\cdot i}}$$

b) $z = -2 + i\cdot 2\cdot\sqrt{3}$



$|z| = 2\cdot\sqrt{1+3} = 4$

$\theta = \tan^{-1}\frac{y}{x} + k\cdot\pi$

$\theta = \tan^{-1}(-\sqrt{3}) + \pi$

$\theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

$$\underline{\underline{z = 4\cdot e^{i\frac{2\pi}{3}}}}$$

1c)

$$z^4 = i$$

$$z^4 = e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}} \cdot e^{ik\cdot 2\pi} \quad ; k \in \mathbb{Z}$$

$$z^4 = e^{i\left(\frac{\pi}{2} + k\cdot 2\pi\right)}$$

$$\left[z^4\right]^{\frac{1}{4}} = \left[e^{i\left(\frac{\pi}{2}+k\cdot 2\pi\right)}\right]^{\frac{1}{4}}$$

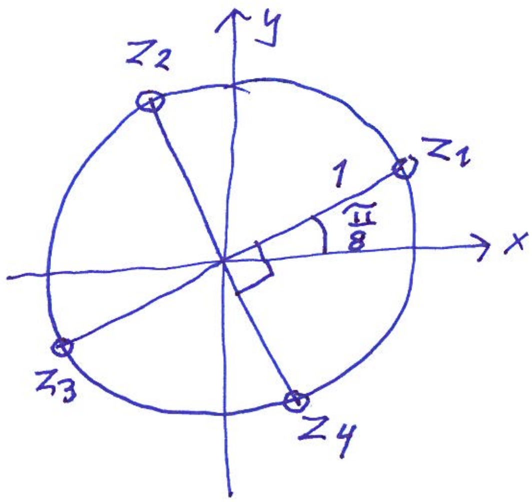
$$\underline{z = e^{i\left(\frac{\pi}{8} + k\cdot\frac{\pi}{2}\right)}} \qquad (|z| = 1)$$

$k=0 \Rightarrow z_1 = e^{i\frac{\pi}{8}}$

$k=1 \Rightarrow z_2 = e^{i\frac{5\pi}{8}}$

$k=2 \Rightarrow z_3 = e^{i\frac{9\pi}{8}}$

$k=3 \Rightarrow z_4 = e^{i\frac{13\pi}{8}}$



Oppgave 2

a)

$$y'' + 2y' + 5y = e^x$$

$$r^2 + 2r + 5 = 0 \Rightarrow \underline{r = -1 \pm 2i}$$

$$y_H = e^{-x}\cdot\left(C_1\cdot\cos 2x + C_2\cdot\sin 2x\right)$$

$$y_P = Ae^x \Rightarrow y_P'' = y_P' = y_P$$

Setter inn i likningen for å finne en partikulær løsning:

$$Ae^x + 2Ae^x + 5Ae^x = e^x$$

$$8A = 1 \Leftrightarrow A = \frac{1}{8}$$

$$y_P = \frac{1}{8}e^x$$

Generell løsning:

$$y = y_P + y_H$$

$$\underline{\underline{y = \frac{1}{8}e^x + e^{-x}(C_1 \cdot \cos 2x + C_2 \cdot \sin 2x)}}$$

<u>2b)</u> $\frac{dy}{dx} = (1+y)\cdot x^3$; $y(0) = 0$

$$\int \frac{dy}{1+y} = \int x^3\,dx \Rightarrow \ln|y+1| = \frac{1}{4}x^4 + \ln C_1$$

$$\ln\frac{|y+1|}{C_1} = \frac{1}{4}x^4 \Leftrightarrow \frac{|y+1|}{C_1} = e^{\frac{1}{4}x^4}$$

$$|y+1| = C_1 e^{\frac{1}{4}x^4} \Leftrightarrow y+1 = \pm C_1 \cdot e^{\frac{1}{4}x^4}$$

$$\underline{y = -1 + C\cdot e^{\frac{1}{4}x^4}} \;;\; y(0) = 1 - C = 0 \Leftrightarrow \underline{C=1}$$

$$\underline{\underline{y = e^{\frac{1}{4}x^4} - 1}}$$

2c, $y' + 2y = 3x^2 \cdot e^{-2x}$; $y(0) = 1$

Integrerende faktor:

$$\mu = e^{\int 2dx} = e^{2x}$$

$$y' + 2y = 3x^2 \cdot e^{-2x} \quad \Big| \cdot e^{2x}$$

$$y' e^{2x} + 2y e^{2x} = 3x^2$$

$$\frac{d}{dx}(y \cdot e^{2x}) = 3x^2$$

$$y \cdot e^{2x} = x^3 + C \quad \Big| \cdot e^{-2x}$$

$$\underline{y = (x^3 + C) \cdot e^{-2x}}$$

$$y(0) = C = 1 \Rightarrow$$

$$\underline{\underline{y = (x^3 + 1) \cdot e^{-2x}}}$$

Oppgave 3

a) $\int (x^{-\frac{4}{5}} - 2x^{\frac{1}{5}})dx = \dfrac{x^{-\frac{4}{5}+1}}{-\frac{4}{5}+1} - 2\cdot\dfrac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} + C =$

$$\underline{\underline{5x^{\frac{1}{5}} - \frac{5}{3}\cdot x^{\frac{6}{5}} + C}}$$

b) $\displaystyle\int \frac{\cos x \cdot dx}{\sqrt{2+\sin x}}$

Substitusjon: $\left.\begin{array}{l} u = 2+\sin x \\ du = \cos x \cdot dx \end{array}\right\} \Rightarrow$

$$\int \frac{du}{\sqrt{u}} = 2\int \frac{du}{2\sqrt{u}} = 2\sqrt{u} + C = \underline{\underline{2\sqrt{2+\sin x} + C}}$$

c) Delbrøk-oppspalting:

$$\frac{x}{(x-2)(x+4)} \equiv \frac{A}{x-2} + \frac{B}{x+4} \quad \Big| \cdot (x-2)(x+4)$$

$$1\cdot x \equiv (A+B)\cdot x + (4A-2B) \Leftrightarrow$$

$$A+B = 1 \wedge 4A-2B = 0 \Leftrightarrow \underline{A = \tfrac{1}{3}} \wedge \underline{B = \tfrac{2}{3}}$$

$$\int \frac{x\cdot dx}{(x-2)(x+4)} = \underline{\underline{\tfrac{1}{3}\ln|x-2| + \tfrac{2}{3}\ln|x+4| + C}}$$

(3 d) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$

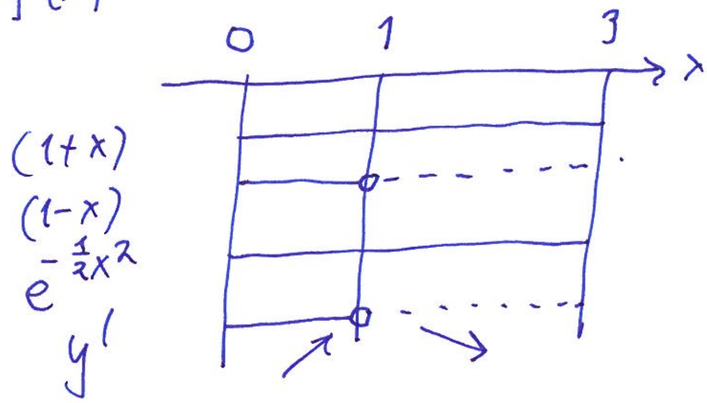
$u = x+2 \Rightarrow du = dx$

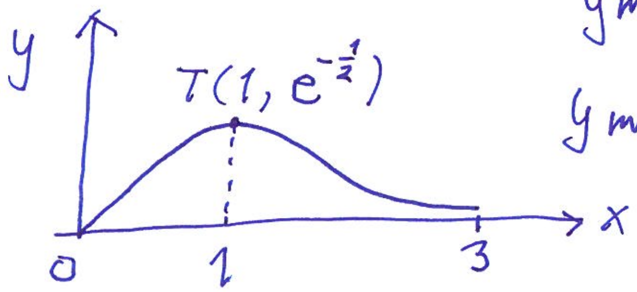
$\displaystyle\int \frac{du}{u^2+1} = \tan^{-1} u + C = \underline{\underline{\tan^{-1}(x+2)+C}}$

Oppgave 4

a) $f(x) = x \cdot e^{-\frac{1}{2}x^2}$; $x \in [0, 3]$

$f'(x) = e^{-\frac{1}{2}x^2} + x \cdot e^{-\frac{1}{2}x^2} \cdot (-x) = (1+x)(1-x) \cdot e^{-\frac{1}{2}x^2}$





$y_{max} = f(1) = \underline{\underline{e^{-\frac{1}{2}}}}$

$y_{min} = f(0) = \underline{\underline{0}}$

strengt voksende : $x \in [0, 1]$

strengt avtagende : $x \in [1, 3]$

4b, (I) $\lim_{x\to 0} \frac{\ln(x+1)}{\sin x} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{\cos x} = \underline{\underline{1}}$

(II) $\lim_{x\to\infty} (\sqrt{x+1} - \sqrt{x}) = \lim_{x\to\infty} \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} =$

$\lim_{x\to\infty} \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = \underline{\underline{0}}$

c) $x^2 + y^2 + 2\cdot\sin(xy-1) = 2$

$P(1,1)$

$\left.\begin{array}{l} V.S. = 1+1+2\sin 0 = \underline{2} \\ H.S. = \underline{2} \end{array}\right\}$ P ligger på kurven.

Implisitt derivasjon:

$2x + 2y\cdot y' + 2\cdot\cos(xy-1)\cdot(y+xy') = 0$

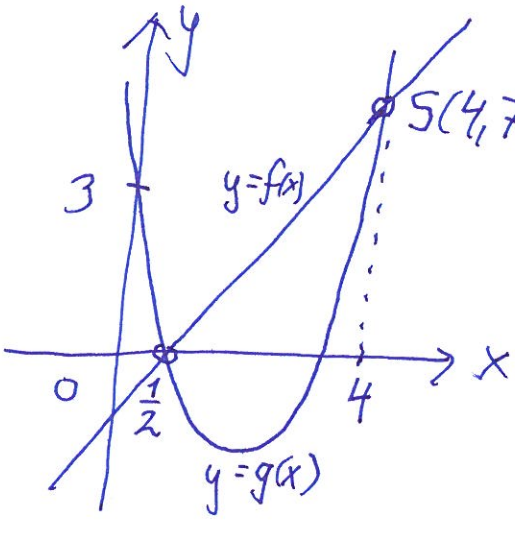
$x + \underline{y\cdot y'} + (y + \underline{x\cdot y'})\cdot\cos(xy-1) = 0$

$y'\cdot[y + x\cdot\cos(xy-1)] = -[x + y\cdot\cos(xy-1)]$

$y' = -\frac{x + y\cdot\cos(xy-1)}{y + x\cdot\cos(xy-1)}$

$\underline{\underline{k}} = \underline{(y')_P} = -\frac{1+1}{1+1} = \underline{\underline{-1}}$

Oppgave 5



$f(x) = 2x - 1$

$g(x) = 2x^2 - 7x + 3$

a) Arealet avgrenset av f og g.

$g(x) = f(x) \Leftrightarrow 2x^2 - 9x + 4 = 0 \Leftrightarrow \underline{x = \frac{1}{2}} \vee \underline{x = 4}$

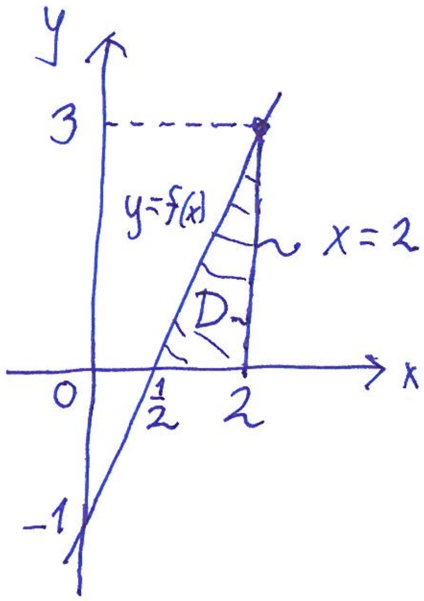
$$\underline{\underline{A}} = \int_{x_1}^{x_2} \{f(x) - g(x)\}\,dx = \int_{\frac{1}{2}}^{4} (-2x^2 + 9x - 4)\,dx =$$

$$\left[-\frac{2}{3}x^3 + \frac{9}{2}x^2 - 4x\right]_{\frac{1}{2}}^{4} =$$

$$\left(-\frac{128}{3} + 72 - 16\right) - \left(-\frac{1}{12} + \frac{9}{8} - 2\right) =$$

$$\frac{40}{3} + \frac{23}{24} = \underline{\underline{\frac{343}{24}}} = \underline{\underline{14\frac{7}{24}}}$$

5b



Volum ved rotasjon om y-aksen.

$$V = \int_{\frac{1}{2}}^{2} \int_{0}^{2x-1} 2\pi x \cdot dy\, dx = 2\pi \int_{\frac{1}{2}}^{2} \int_{0}^{2x-1} dy \cdot x \cdot dx$$

$$= 2\pi \int_{\frac{1}{2}}^{2} (2x-1) \cdot x \cdot dx = 2\pi \int_{\frac{1}{2}}^{2} (2x^2 - x)\, dx$$

$$= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{2}x^2 \right]_{\frac{1}{2}}^{2}$$

$$= 2\pi \cdot \left\{ \left(\frac{16}{3} - 2 \right) - \left(\frac{1}{12} - \frac{1}{8} \right) \right\}$$

$$= \underline{\underline{\frac{27}{4}\pi}}$$

Oppgave 6

$t = t_1 \Rightarrow V = V_1 = 4100\ cm^3$

$V = \frac{4}{3}\pi r^3 \Leftrightarrow \frac{3V}{4\pi} = r^3 \Leftrightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$

$r_1 = \sqrt[3]{\frac{3V_1}{4\pi}} = 9{,}93\ cm$

$\left(\frac{dV}{dt}\right)_{t=t_1} = -3\ cm^3/min = \dot{V}_1$

$S = 4\pi r^2 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \cdot \dot{r} \Rightarrow$

$\dot{V} = S \cdot \dot{r} \Rightarrow \dot{V}_1 = S_1 \cdot \dot{r}_1 \Leftrightarrow$

$\dot{r}_1 = \frac{\dot{V}_1}{S_1} = \frac{\dot{V}_1}{4\pi r_1^2} = \frac{-3\ cm^3/min}{1239\ cm^2} = -0{,}00242\ cm/min$

$\frac{dS}{dt} = \dot{S} = \frac{dS}{dr} \cdot \frac{dr}{dt} = 8\pi r \cdot \dot{r}$

$\left(\frac{dS}{dt}\right)_{t=t_1} = \dot{S}_1 = 8\pi r_1 \cdot \dot{r}_1 = \frac{8\pi r_1 \cdot \dot{V}_1}{4\pi r_1^2} = \frac{2\dot{V}_1}{r_1}$

$\dot{S}_1 = \frac{2 \cdot (-3\ cm^3/min)}{9{,}93\ cm} = -0{,}604\ cm^2/min$

6b)

$$\frac{dV}{dt} = k \cdot S \quad ; \qquad V = \frac{4}{3}\pi r^3$$

$$\Updownarrow \qquad\qquad S = \frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dr} \cdot \frac{dr}{dt} = k \cdot S$$

$$S \cdot \dot{r} = k \cdot S$$

$$\dot{r} = k \Rightarrow \underline{r = r_0 + k \cdot t}$$

$$r_1 = r(t_1) = r_0 + k \cdot t_1 \Rightarrow \dot{r}_1 = k$$

$$r = r(t) = r_0 + \dot{r}_1 t$$

$$r_2 = r(t_2) = r_0 + \dot{r}_1 t_2 = 0 \iff$$

$$t_2 = \frac{r_0}{-\dot{r}_1} = \frac{r_1 - \dot{r}_1 t_1}{-\dot{r}_1} = t_1 + \frac{r_1}{-\dot{r}_1}$$

$$\underline{\underline{t_2 - t_1}} = \frac{r_1}{-\dot{r}_1} = \frac{9{,}93 \text{ cm}}{0{,}00242 \text{ cm/min}}$$

$$= \underline{4100 \text{ min}}$$

Eller:

$$\underline{\underline{t_2 - t_1}} = \frac{3V_1}{-\dot{V}_1} = \underline{\frac{3V_1}{3}} = \underline{\underline{4100 \text{ min}}} = \underline{\underline{68 \text{h } 20 \text{ min}}}$$