Ex. 1.2

$$\frac{dU}{dt} = -\alpha U \quad , \alpha > 0$$

$$U(t=0) = U_0$$

$$\int_{U_0}^{U} \frac{1}{U}\, dU = -\int_0^t \alpha \, dt$$

$$e^{\ln U - \ln U_0} = e^{-\alpha t}$$

$$\frac{U}{U_0} = e^{-\alpha t}$$

$$U(t) = U_0 e^{-\alpha t} = \frac{U_0}{e^{\alpha t}} \quad , t > 0$$

Perturbation in U_0:

consider $\frac{dU}{dt} = -\alpha U$, $U(t=0) = [U_0 + \varepsilon]$, ε small

$$\Rightarrow U(t) = [U_0 + \varepsilon] e^{-\alpha t} = \frac{U_0 + \varepsilon}{e^{\alpha t}} \stackrel{def}{=} U^{\varepsilon}$$

Stability of solutions: $U - U^{\varepsilon} = \frac{U_0}{e^{\alpha t}} - \frac{U_0 + \varepsilon}{e^{\alpha t}} = \frac{-1\,\varepsilon}{e^{\alpha t}}$

$$\Rightarrow |U - U^{\varepsilon}| \leq \frac{\varepsilon}{e^{\alpha t}} \leq \varepsilon \, , \, t > 0 \; (\text{since } e^{\alpha t} \geq 1 \text{ for } t \geq 0)$$

Conclusion:

This problem is stable in the sense that $|(U - U^{\varepsilon})(t)|$ for any time $t > 0$ is controlled by ε, the initial disturbance

Ex. 1.3

$\circledast \quad \frac{du}{dt} = tu(u-2), \quad u(t=0) = u_0$

$\Rightarrow \quad u(t) = \frac{2u_0}{u_0 + [2-u_0]e^{t^2}}, \quad t > 0$

a) check solution:

$t=0: \quad u(t=0) = \frac{2u_0}{u_0 + [2-u_0]\cdot e^0} = \frac{2u_0}{u_0+2-u_0} = \frac{2u_0}{2} = u_0$ (ok)

$t>0$:

LHS of $\circledast = \frac{du}{dt} = 2u_0 \cdot \frac{d}{dt}\left(\frac{1}{u_0 + [2-u_0]e^{t^2}}\right)$

$= 2u_0 \cdot \frac{-1}{(u_0 + [2-u_0]e^{t^2})^2} \cdot [2-u_0] 2t e^{t^2}$

$= -\frac{4u_0 t e^{t^2}[2-u_0]}{(u_0 + [2-u_0]e^{t^2})^2}$

RHS of $\circledast = tu(u-2) = t \cdot \frac{2u_0}{u_0 + [2-u_0]e^{t^2}}\left(\frac{2u_0}{u_0 + [2-u_0]e^{t^2}} - 2\right)$

$= \frac{4u_0^2 t}{(u_0 + [2-u_0]e^{t^2})^2} - \frac{4tu_0}{u_0 + [2-u_0]e^{t^2}}$

$= \frac{4u_0^2 t - 4tu_0(u_0 + [2-u_0]e^{t^2})}{(u_0 + [2-u_0]e^{t^2})^2} = -\frac{4tu_0[2-u_0]e^{t^2}}{(u_0 + [2-u_0]e^{t^2})^2}$

Conclusion: LHS of $\circledast$ = RHS of $\circledast$

b) $0 \le u_0 \le 2$ $\quad u(t) = \frac{2u_0}{u_0 + [2-u_0]e^{t^2}} \le \frac{2u_0}{u_0 + 0} = 2$, since $(2-u_0)e^{t^2} \ge 0, t \ge 0$ and $u_0 \in [0,2]$

Upper limit: $u(t) \le 2$

Lower limit:

$u_0 = 0 \Rightarrow u(t) = 0 \Rightarrow u(t) \ge 0$

$u_0 = 2 \Rightarrow u(t) = \frac{2u_0}{u_0} = 2 > 0$

$0 < u_0 < 2 \Rightarrow u(t) = \frac{2u_0}{u_0 + [2-u_0]e^{t^2}} > 0$

$\Rightarrow u(t) \ge 0$

Conclusion: $u(t) \in [0,2]$

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c) assume $U_0 > 2$:

$$U(t) = \frac{2U_0}{U_0 + (2-U_0)e^{t^2}} = \frac{2U_0}{U_0 - (U_0-2)e^{t^2}} \quad (*)$$

clearly, for small $t>0$ $U_0 > (U_0-2)e^{t^2}$,

however, as t increases $(U_0-2)e^{t^2}$ will approach U_0.

at some time $t = t^*$ given by

$$(U_0-2)e^{(t^*)^2} = U_0$$, $(*)$ will blow up as the denominator becomes 0

$$e^{(t^*)^2} = \frac{U_0}{U_0-2}$$

$$(t^*)^2 = \ln\left(\frac{U_0}{U_0-2}\right)$$

$$t^* = \left(\ln \frac{U_0}{U_0-2}\right)^{\frac{1}{2}}$$

d) $U_0 \in [0.9, 1.1]$

$U(t) = \frac{2U_0}{U_0 + (2-U_0)e^{t^2}}$ stays close to $\frac{2}{1+e^{t^2}}$ which will approach 0 as $t \to \infty$

This behavior is the same for all $U_0 \in [0.9, 1.1]$

Conclusion: $U(t)$ is stable for such data U_0

Ex. 1.5

a) $U_t + 2x\,U_x = 0$

$U(x,t=0) = e^{-x^2} = \phi(x)$

Characteristics: $\dfrac{dX(t)}{dt} = 2X, \quad X(t=0) = X_0$

$$\frac{1}{X}\,dX = 2dt$$

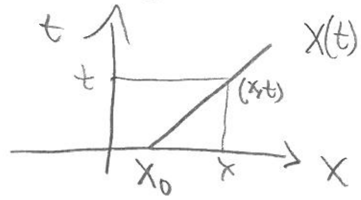
$$\ln X = 2t + C$$

$$e \qquad e$$

$$X(t) = \tilde{C} e^{2t}$$

$$X(t) = X_0 e^{2t}$$

Solution: $\dfrac{dU(X(t),t)}{dt} = 0 \Rightarrow U(X(t),t) = U(X_0, t=0) = \phi(X_0)$



$$U(x,t) = U(X(t),t) = \phi(X_0), \quad X_0 = X(t)e^{-2t}$$

$$= \phi\left(X(t)e^{-2t}\right)$$

$$= \phi\left(xe^{-2t}\right) = e^{-[x^2 e^{-4t}]}$$

Check: $\quad t=0$: $U(x,t=0) = e^{-[x^2 \cdot 1]} = e^{-x^2} = \phi(x)$ (ok)

$t>0$: $U_t = \dfrac{\partial}{\partial t}\phi\left(xe^{-2t}\right) = \phi' \dfrac{\partial}{\partial t}\left(xe^{-2t}\right)$

$$= \phi' x(-2)e^{-2t}$$

$$U_x = \frac{\partial}{\partial x}\phi\left(xe^{-2t}\right) = \phi' \frac{\partial}{\partial x}\left(xe^{-2t}\right)$$

$$= \phi' e^{-2t}$$

Conclusion: $U_t + 2xU_x$

$$= -2xe^{-2t}\phi' + 2x \cdot e^{-2t}\phi' = 0 \text{ (ok)}$$

b) $U_t - xU_x = 0$

$U(x, t=0) = \sin(87x) = \phi(x)$

Characteristics: $\frac{dX(t)}{dt} = -X$, $X(t=0) = X_0$

$$\int \frac{1}{X} dX = -\int dt$$

$$e^{\ln X} = e^{-t+c}$$

$$X(t) = \tilde{c} e^{-t}$$

$$X(t) = X_0 e^{-t}$$

Solution: $\frac{dU(X(t),t)}{dt} = 0 \Rightarrow U(X(t),t) = \phi(X_0)$, $\quad X_0 = X(t)e^t$

$$U(x,t) = \phi(X_0) = \phi(xe^t) = \sin(87xe^t)$$

Check:

$t=0$: $U(x,t=0) = \sin(87x)$ (ok)

$t>0$: $U_t = \phi' \frac{\partial}{\partial t}(xe^t) = xe^t \phi'$, $\quad U_x = \phi' \frac{\partial}{\partial x}(xe^t) = e^t \phi'$

$\Rightarrow U_t - xU_x = xe^t\phi' - xe^t\phi' = 0$ (ok)

c) $U_t + xU_x = x$

$U(x,0) = \cos(90x) = \phi(x)$

Characteristics: $\frac{dX(t)}{dt} = X$, $X(t=0) = X_0$

$$X(t) = X_0 e^t$$

Solution: $U(X(t),t)$

$$\frac{dU(X(t),t)}{dt} = U_x \frac{dX}{dt} + U_t \frac{dt}{dt}$$

$$= U_x X + U_t = X(t)$$

$$\int_{U(X_0,t=0)}^{U(X(t),t)} dU(X(t),t) = \int_0^t X(t)\, dt$$

$$U(X(t),t) - U(X_0,t=0) = \int_0^t X_0 e^t\, dt = X_0 e^t\Big|_0^t = X_0(e^t - 1)$$

$$U(X(t),t) = \phi(X_0) + X_0(e^t - 1), \quad X_0 = X(t)e^{-t}$$

Conclusion:

$$U(x,t) = \phi(xe^{-t}) + xe^{-t}(e^t - 1)$$

$$= \phi(xe^{-t}) + x(1 - e^{-t})$$

$$= \cos(90xe^{-t}) + x(1 - e^{-t})$$

Check:

$t=0$: $U(x,t=0) = \cos(90x)$ (ok)

$t>0$: $U_t = \phi' \cdot [-xe^{-t}] + xe^{-t}$

$U_x = \phi' \cdot e^{-t} + (1 - e^{-t})$

Hence, $U_t + xU_x = -xe^{-t}\phi' + xe^{-t} + xe^{-t}\phi' + x(1 - e^{-t})$

$= x$

(ok)

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d) $U_t + x U_x = x^2$

$$U(x, t=0) = \sin(87x)\cos(90x) \overset{\text{def}}{=} \phi(x)$$

<u>characteristics:</u> $\frac{dX(t)}{dt} = X(t), \quad X(t=0) = x_0$

$$X(t) = x_0 e^{t}$$

Consider $U(X(t), t)$:

$$\frac{d}{dt} U(X(t), t) = U_x \frac{dX}{dt} + U_t = U_x X(t) + U_t = X(t)^2$$

$$\int_{U(x_0,0)}^{U(X(t),t)} dU = \int_0^t X(t)^2 \, dt$$

$$U(X(t), t) - \phi(x_0) = \int_0^t x_0^2 e^{2t} \, dt = x_0^2 \frac{1}{2} e^{2t}\Big|_0^t = \frac{1}{2} x_0^2 \left(e^{2t} - 1\right)$$

$$U(X(t), t) = \phi(x_0) + \frac{1}{2} x_0^2 \left(e^{2t} - 1\right), \quad x_0 = x e^{-t}$$

$$\Longrightarrow \quad U(x, t) = \phi(x e^{-t}) + \frac{1}{2} x^2 e^{-2t} \left(e^{2t} - 1\right)$$

$$= \phi(x e^{-t}) + \frac{1}{2} x^2 \left(1 - e^{-2t}\right)$$

<u>check:</u>

$$U(x, t=0) = \phi(x) \quad \text{ok}$$

$$\left.\begin{aligned} U_t &= \phi' \cdot (-x e^{-t}) + \frac{1}{2} x^2 \cdot 2 e^{-2t} \\ U_x &= \phi' \cdot (e^{-t}) + x(1 - e^{-2t}) \end{aligned}\right\} \quad \begin{aligned} &U_t + x U_x \\ &= \phi' \cdot (-x e^{-t}) + x^2 e^{-2t} \\ &\quad + x e^{-t} \phi' + x^2 (1 - e^{-2t}) \\ &= x^2 \quad \text{ok} \end{aligned}$$

(Plot some solutions in matlab !)