Ex. 1.6 $U_t + U_x = U$

$U(x,0) = \phi(x)$

characteristics: $\frac{dX(t)}{dt} = 1$, $X(t=0) = X_0$

$\Rightarrow X(t) = t + X_0$

Solution: Consider $U(X(t),t)$

$$\frac{dU(X(t),t)}{dt} = U_x \frac{dX}{dt} + U_t \frac{dt}{dt} = U_x + U_t = U(X(t),t)$$

$$\int_{U(X_0,t=0)}^{U(X(t),t)} \frac{1}{U}\, dU(X(t),t) = \int_0^t dt$$

$$\ln U(X(t),t) - \ln \phi(X_0) = t$$

$$e^{\ln \frac{U(X(t),t)}{\phi(X_0)}} = e^t$$

$$U(X(t),t) = \phi(X_0)e^t, \quad X_0 = X(t) - t$$
$$= \phi(X(t)-t)e^t$$

Hence, $U(x,t) = \phi(x-t)e^t$ → growth

translation/transport

Check: → $t=0$: $U(x,t=0) = \phi(x)$ (ok)

→ $t>0$ $U_t = \phi' \frac{\partial}{\partial t}(x-t)\cdot e^t + \phi(x-t)\frac{\partial}{\partial t}e^t$

$= -\phi' e^t + \phi(x-t)e^t$

$U_x = \phi' \cdot e^t$

$$\Rightarrow U_t + U_x = -\phi' e^t + \phi(x-t)e^t + \phi' e^t = \phi(x-t)e^t = U$$ ok

alternatively:

$W = U - V$

$W_t + aW_x = 0$

$W(x,t=0) = -\varepsilon(x)$

$\Rightarrow w(x,t) = -\varepsilon(X_0)$ where $X = at + X_0$

B7

Ex. 1.7

Stability of nonhomogeneous eq.

Consider
(i) $U_t + aU_x = b(x,t)$, a constant
$U(x,t=0) = \phi(x)$
(ii) $V_t + aV_x = b(x,t)$
$V(x,t=0) = \phi(x) + \varepsilon(x)$

Solutions:

(i) $\frac{dX(t)}{dt} = a$, $X(t=0) = X_0$

$X(t) = at + X_0$

$$\frac{dU(X(t),t)}{dt} = U_x a + U_t = b(X(t),t)$$

$$\Rightarrow \int_{\phi(X_0)}^{U} dU(X(t),t) = \int_0^t b(X(t),t)\,dt$$

$$U(X(t),t) = \phi(X_0) + \int_0^t b(X(t),t)\,dt = \phi(X_0) + \int_0^t b(at+X_0,t)\,dt$$

(ii) Similarly,

$$V(x,t) = \phi(X_0) + \varepsilon(X_0) + \int_0^t b(at+X_0,t)\,dt$$

⇓

$$U(x,t) - V(x,t) = -\varepsilon(X_0)$$

$$|U(x,t) - V(x,t)| = |\varepsilon(X_0)| \leq \sup_{x\in\mathbb{R}} |\varepsilon(x)|$$

⇓

$$\sup_{x\in\mathbb{R}, t\geq 0} |U(x,t) - V(x,t)| \leq \sup_{x\in\mathbb{R}} |\varepsilon(x)|$$

difference for all times ≤ initial difference

Exercise 1.10

B8

(*) $V_t = V_{xx}$

$V(x,t=0) = V_0(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases} \Rightarrow V(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$

Consider: $U_t = \varepsilon U_{xx}$, $\varepsilon > 0$ (**)

$U(x,t=0) = V_0(x)$ (as above)

Find solution of (**)

(How to relate model (**) to (*)?)

Rescale time variable t:

$$\hat{t} = t\varepsilon$$

and consider $U(x,t) = U(x, \frac{\hat{t}}{\varepsilon}) \stackrel{\text{def}}{=} \tilde{U}(x,\hat{t})$. What is the equation for $\tilde{U}(x,\hat{t})$?

$$\Rightarrow \left\{ \begin{array}{l} \frac{\partial \tilde{U}}{\partial \hat{t}} = \frac{\partial U}{\partial t} \cdot \frac{\partial t}{\partial \hat{t}} = \boxed{\frac{\partial U}{\partial t} \cdot \frac{1}{\varepsilon} = U_{xx}} = \tilde{U}_{xx} \\ \tilde{U}(x,\hat{t}=0) = U(x,t=0) = V_0(x) \end{array} \right\}$$

$U(x,t)$ solves (**) $\Updownarrow$ $\tilde{U}(x,\hat{t})$ solves (*)

Conclusion: $\tilde{U}(x,\hat{t}) = U(x,t)$ satisfies (*) when $\hat{t}$ is timescale

Solution: $\tilde{U}(x,\hat{t}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{\hat{t}}}} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t\varepsilon}}} e^{-\theta^2} d\theta = U(x,t)$

$U(x,t)$ should then be a solution of (**).

How will different values of $\varepsilon > 0$ affect the solution $U(x,t)$?

Exercise 1.13 (B9)

Consider $U(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$ ⊛

a) Does ⊛ satisfy $U_t = U_{xx}$?

$$U_t = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4t}} \cdot \frac{\partial}{\partial t}\left(\frac{x}{2\sqrt{t}}\right) = \frac{x}{2\sqrt{\pi}} \cdot \left(-\frac{1}{2}\right) t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}}$$

$$U_x = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4t}} \frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{t}}\right) = \frac{1}{2\sqrt{\pi}\sqrt{t}} e^{-\frac{x^2}{4t}}$$

$$U_{xx} = \frac{1}{2\sqrt{\pi t}} \cdot \frac{\partial}{\partial x}\left(e^{-\frac{x^2}{4t}}\right)$$

$$= \frac{1}{\cancel{2}\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \cdot \frac{-\cancel{2}x}{4t} = -\frac{1}{4\sqrt{\pi}\, t^{3/2}} x e^{-\frac{x^2}{4t}}$$

Note that

$$\frac{d}{dt}\int_a^{b(t)} F'(s)\, ds = \frac{d}{dt}\left(F(b(t)) - F(a)\right) = F'(b(t)) \cdot b'(t)$$

Conclusion: $U_t = U_{xx}$, $x \in \mathbb{R}$, $t > 0$

b) Let $t > 0$.

Is $U(\cdot,t) \in C^\infty(\mathbb{R})$ (U is smooth)?

$\Updownarrow$

$U_x(\cdot,t)$ is continous

$U_{xx}(\cdot,t)$ — " —

$U_{xxx}(\cdot,t)$ — " —

⋮

$U_x = \frac{1}{2\sqrt{\pi t}} = e^{-\frac{x^2}{4t}}$ is continous in x

$U_{xx} = -\frac{x}{4\sqrt{\pi}\, t^{3/2}} e^{-\frac{x^2}{4t}}$ continous in x

$$U_{xxx} = -\frac{1}{4\sqrt{\pi}\, t^{3/2}} e^{-\frac{x^2}{4t}} - \frac{x}{4\sqrt{\pi}\, t^{3/2}} \left(e^{-\frac{x^2}{4t}}\right)_x$$

$$= -\frac{1}{4\sqrt{\pi}\, t^{3/2}} e^{-\frac{x^2}{4t}} + \frac{x}{4\sqrt{\pi}\, t^{3/2}} \cdot e^{-\frac{x^2}{4t}} \cdot \frac{2x}{4t},$$

continous in x

Remark: Instantly, the initial jump at $x=0$ is changed into a smooth function!

c) $U(0,t) = \frac{1}{2}$?

$$U(x=0,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \sqrt{\pi} \frac{1}{2} = \frac{1}{2}, \quad \text{since } \int_{-\infty}^{\infty} e^{-\theta^2} d\theta = \sqrt{\pi} = 2\int_{-\infty}^{0} e^{-\theta^2} d\theta$$

(See Exercise 1.11)

d) $x \neq 0$

$$U(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$$

Conclusion: $U(x,t) \longrightarrow H(x)$, when $t \to 0^+$

$x < 0$: $\frac{x}{2\sqrt{t}} \to -\infty$ when $t \to 0^+ \Rightarrow U(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\infty} e^{-\theta^2} d\theta = 0$

$x > 0$: $\frac{x}{2\sqrt{t}} \to +\infty$ when $t \to 0^+ \Rightarrow U(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$