Solution Pet 510 Fall 2013: Part A

Problem 1.

(a) Consider the linear transport equation

$$(*) \qquad u_t + a(x)u_x = b(x,u), \qquad x \in \mathbb{R} = (-\infty, +\infty), \qquad u(x,t=0) = u_0(x) = \phi(x).$$

- Transport effect represented by the term $a(x)u_x$: If $a(x) < 0$, then a transport from right towards left. If $a(x) > 0$, transport from left towards right.
Impact from the source term $b(x,u)$: If $b > 0$ this will lead to a growth of u, whereas $b < 0$ will imply that u is reduced?

(b) Let $a(x) = x$ and $b(x,u) = 0$ in (*). Characteristic $X(t)$ is given

$$\frac{d}{dt}X(t) = X(t), \qquad X(t=0) = x_0,$$

which gives the solution $X(t) = x_0 e^t$. Moreover,

$$\frac{d}{dt}u(X(t),t) = u_x \frac{dX}{dt} + u_t = u_x X(t) + u_t = 0,$$

since u is a solution of $u_t + xu_x = 0$ and satisfies $u_t + X(t)u_x = 0$ along $X(t)$, i.e.

$$u(X(t),t) = u(x_0, t=0) = \phi(x_0) = \phi(X(t)e^{-t}).$$

Conclusion: $u(x,t) = \phi(xe^{-t})$.
Check:
(i) We see that $u(x,t=0) = \phi(xe^0) = \phi(x)$, thus, initial data is satisfied.
(ii) Moreover, we see that

$$u_t = \phi'(xe^{-t})(xe^{-t})_t = \phi'(xe^{-t})x \cdot (-1) \cdot e^{-t}$$

and

$$u_x = \phi'(xe^{-t})(xe^{-t})_x = \phi'(xe^{-t})e^{-t}$$

so, clearly, $u_t + xu_x = 0$.

(c) Let $a(x) = x$ and $b(x,u) = u$ in (*). Characteristic $X(t)$ is given as above. Moreover,

$$\frac{d}{dt}u(X(t),t) = u_x \frac{dX}{dt} + u_t = u_x X(t) + u_t = u(X(t),t),$$

since u satisfies $u_t + xu_x = u$. From this we get

$$\int_{u(x_0,t=0)}^{u(X(t),t)} \frac{1}{u}\,du = \int_0^t dt$$

which gives us

$$\ln(u(X(t),t)) - \ln(\phi(x_0)) = t, \qquad \text{or} \qquad u(X(t),t) = \phi(x_0)e^t = \phi(X(t)e^{-t})e^t.$$

Conclusion: $u(x,t) = \phi(xe^{-t})e^t$
Check:

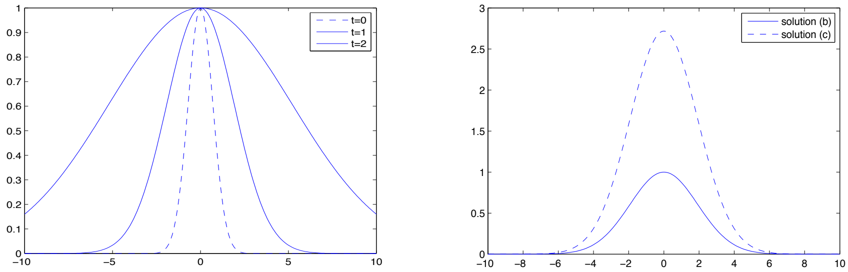

FIGURE 1. **Left:** Plot of u in (b) at times $t = 1$ and $t = 2$. **Right:** Comparison of u from (b) and (c) computed at time $t = 1$.

(i) We see that $u(x, t = 0) = \phi(xe^0)e^0 = \phi(x)$, thus, initial data is satisfied.
(ii) Moreover, we see that

$$\begin{aligned} u_t &= \phi'(xe^{-t})(xe^{-t})_t \cdot e^t + \phi(xe^{-t}) \cdot (e^t)_t \\ &= -\phi'(xe^{-t})xe^{-t} \cdot e^t + \phi(xe^{-t})e^t = -\phi'(xe^{-t})x + \phi(xe^{-t})e^t \end{aligned}$$

and

$$u_x = \phi'(xe^{-t})(xe^{-t})_x \cdot e^t = \phi'(xe^{-t})e^{-t} \cdot e^t = \phi'(xe^{-t})$$

Clearly, $u_t + xu_x = \phi(xe^{-t})e^t = u$

(d) Now we choose $\phi(x) = \exp(-x^2)$.
- Solution of (b) at times $t = 1$ and $t = 2$ is shown in Fig. 1 (left).
- Solution of (b) and (c) at time $t = 1$ are shown in Fig. 1 (right). Main difference is the growth in u for solution computed in part (c).

(e) $U^n_{j+1/2} = u^n_j$ and $U^n_{j-1/2} = u^n_{j-1}$. This gives the scheme $u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}(u^n_j - u^n_{j-1})$.

(f) Estimate for the discrete scheme:

$$|u^{n+1}_j| = |u^n_j(1-\lambda) + \lambda u^n_{j-1}| \leq |u^n_j(1-\lambda)| + |\lambda u^n_{j-1}| = (1-\lambda)|u^n_j| + \lambda|u^n_{j-1}|$$

Here we first have used the triangle inequality $|a+b| \leq |a|+|b|$. Then we have assumed that $0 \leq \lambda \leq 1$ in order to ensure that $(1-\lambda) \geq 0$ and $\lambda \geq 0$. Now we sum over all cells from 1 to M

$$\begin{aligned}
\sum_{j=1}^{M} |u_j^{n+1}| &\leq (1-\lambda)\sum_{j=1}^{M} |u_j^n| + \lambda \sum_{j=1}^{M} |u_{j-1}^n| \\
&= (1-\lambda)\sum_{j=1}^{M} |u_j^n| + \lambda \sum_{j=0}^{M-1} |u_j^n| \qquad \text{(shift of index in last sum)} \\
&\leq (1-\lambda)\sum_{j=1}^{M} |u_j^n| + \lambda \sum_{j=1}^{M} |u_j^n| \qquad \text{(add } |u_M^n| \text{ in the last sum)} \\
&= \sum_{j=1}^{M} |u_j^n|.
\end{aligned}$$

Hence, the conclusion is that $\sum_{j=1}^{M} |u_j^{n+1}| \leq \sum_{j=1}^{M} |u_j^n| \leq \ldots \leq \sum_{j=1}^{M} |u_j^0|$,
Condition on the discretization parameters Δt and Δx: $0 \leq \lambda = \frac{\Delta t}{\Delta x} \leq 1$

<u>Problem 2</u>.

(a) Mass balance

$$(\phi\rho)_t + (\rho u)_x = 0,$$

where ϕ, ρ, and u are porosity, fluid density, and fluid velocity (Darcy velocity).
Darcy's law:

$$u = -\frac{k}{\mu} p_x$$

This gives

$$(\phi(p)\rho(p))_t = (\frac{k}{\mu}\rho(p)p_x) = \frac{k}{\mu}(\rho(p)p_x)_x$$

Using assumptions on ϕ and ρ we get

$$\rho\phi_0[1 + c_r(p - p_0)]_t = \rho\phi_0 c_r p_t = \frac{k}{\mu}\rho p_{xx}.$$

This gives us

$$p_t = \kappa p_{xx}, \qquad \kappa = \frac{k}{\mu\phi_0 c_r}$$

(b) Verify that

$$p(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$$

is a solution of $p_t = p_{xx}$. Note that $F(x,t) = \int_{-\infty}^{b(x,t)} G'(\theta)\theta = G(b(x,t)) - G(-\infty)$.
Consequently,

$$F(x,t)_t = G'(b)b_t, \qquad F(x,t)_x = G'(b)b_x.$$

Using this with $b = \frac{x}{2\sqrt{t}}$ and $G'(\theta) = e^{-\theta^2}$ we get

$$p_t = \frac{1}{\sqrt{\pi}} e^{-b^2} b_t = \frac{1}{\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \left[\frac{x}{2\sqrt{t}}\right]_t = \frac{x}{2\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \cdot \frac{-1}{2} t^{-3/2}$$

and

$$p_x = \frac{1}{\sqrt{\pi}} e^{-b^2} b_x = \frac{1}{\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \left[\frac{x}{2\sqrt{t}}\right]_x = \frac{1}{2\sqrt{\pi} t^{1/2}} e^{-[\frac{x}{2\sqrt{t}}]^2}$$

and

$$p_{xx} = \frac{1}{2\sqrt{\pi} t^{1/2}} \left[e^{-[\frac{x^2}{4t}]}\right]_x = -\frac{1}{2\sqrt{\pi} t^{1/2}} e^{-[\frac{x^2}{4t}]} \cdot [\frac{x^2}{4t}]_x == -\frac{x}{4\sqrt{\pi} t^{3/2}} e^{-[\frac{x^2}{4t}]}$$

Clearly, $p_t = p_{xx}$.
Initial condition:

$$x < 0: \qquad t \to 0^+ \Rightarrow \frac{x}{2\sqrt{t}} \to -\infty \Rightarrow p(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\infty} e^{-\theta^2} d\theta = 0$$

and

$$x > 0: \qquad t \to 0^+ \Rightarrow \frac{x}{2\sqrt{t}} \to +\infty \Rightarrow p(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\theta^2} d\theta = 1$$

(c) Discrete scheme (explicit in time):

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} = \frac{1}{\Delta x}([p_x]_{j+1/2}^n - [p_x]_{j-1/2}^n)$$

$$j = 1, \dots, M-1: \qquad [p_x]_{j+1/2} = \frac{p_{j+1} - p_j}{\Delta x}$$

$$x = 0: \qquad [p_x]_{1/2} = \frac{p_1 - 0}{\Delta x/2}$$

$$x = 1: \qquad [p_x]_{M+1/2} = \frac{0 - p_M}{\Delta x/2}$$

Stability condition: $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$

(d) Estimate: We multiply the equation by p and get

$$pp_t = p_{xx} p$$

or

$$\frac{1}{2}(p^2)_t = (p_x p)_x - p_x p_x.$$

Next, we integrate in space from 0 to 1:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 p^2\,dx = \int_0^1 (p_x p)_x\,dx - \int_0^1 (p_x)^2\,dx.$$

For the first term on the right hand side we get:

$$\int_0^1 (p_x p)_x\,dx = p_x p|_{x=0}^{x=1} = 0,$$

using the boundary condition. Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 p^2\,dx = -\int_0^1 (p_x)^2\,dx \le 0$$

which gives the inequality $\frac{d}{dt}\int_0^1 p^2\,dx \le 0$. We finally integrate in time over $[0,t]$.