EXERCISES 10.2

1. Let $A = (-1, 2)$, $B = (2, 0)$, $C = (1, -3)$, $D = (0, 4)$. Express each of the following vectors as a linear combination of the standard basis vectors **i** and **j** in $\mathbb{R}^2$.

(a) $\overrightarrow{AB}$, (b) $\overrightarrow{BA}$, (c) $\overrightarrow{AC}$, (d) $\overrightarrow{BD}$, (e) $\overrightarrow{DA}$,

(f) $\overrightarrow{AB} - \overrightarrow{BC}$, (g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD}$, and

(h) $\dfrac{\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}}{3}$.

In Exercises 2–3, calculate the following for the given vectors **u** and **v**:

(a) $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} - 3\mathbf{v}$,

(b) the lengths $|\mathbf{u}|$ and $|\mathbf{v}|$,

(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

4. Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

7. The diagonals of any parallelogram bisect each other.

8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B? For what values of k is the trip not possible?

12. A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?

13. For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

14. Find the angle between a diagonal of a cube and one of the edges of the cube.

15. Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers.

16. **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

17. Find a unit vector that makes equal angles with the three coordinate axes.

18. Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

19. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20. Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

21. Let $\mathbf{a}$ be a nonzero vector, and let b be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

22. Find two unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

23. Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

24. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

26. Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are arbitrary real numbers.

27. **(The triangle inequality)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

28. (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

29. **(Orthonormal bases)** Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

30. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

31. **(Resolving a vector in perpendicular directions)** If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

32. **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}$ are position vectors of points U, V, and P, respectively, that $\mathbf{u}$ is not parallel to $\mathbf{v}$, and that P lies in the plane containing the origin, U, and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both $\mathbf{v}$ and $\mathbf{r}$ as sums of vectors parallel and perpendicular to $\mathbf{u}$ as suggested in Exercise 31.

33. Given constants r, s, and t, with $r \neq 0$ and $s \neq 0$, and given a vector $\mathbf{a}$ satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

Hanging cables

34. **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is $\delta g x$ rather than $\delta g s$, show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

EXERCISES 10.3

1. Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

2. Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$.

3. Find the area of the triangle with vertices $(1, 2, 0)$, $(1, 0, 2)$, and $(0, 3, 1)$.

4. Find a unit vector perpendicular to the plane containing the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. What is the area of the triangle with these vertices?

5. Find a unit vector perpendicular to the vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$.

6. Find a unit vector with positive $\mathbf{k}$ component that is perpendicular to both $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Verify the identities in Exercises 7–11, either by using the definition of cross product or the properties of determinants.

7. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

8. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

9. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

10. $(t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$

11. $\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$

12. Obtain the addition formula

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

by examining the cross product of the two unit vectors $\mathbf{u} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$ and $\mathbf{v} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$. Assume $0 \le \alpha - \beta \le \pi$. *Hint:* Regard $\mathbf{u}$ and $\mathbf{v}$ as position vectors. What is the area of the parallelogram they span?

13. If $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$, show that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}$.

14. **(Volume of a tetrahedron)** A **tetrahedron** is a pyramid with a triangular base and three other triangular faces. It has four vertices and six edges. Like any pyramid or cone, its volume is equal to $\frac{1}{3}Ah$, where A is the area of the base and h is the height measured perpendicular to the base. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors coinciding with the three edges of a tetrahedron that meet at one vertex, show that the tetrahedron has volume given by

$$\text{Volume} = \frac{1}{6}|\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})| = \frac{1}{6}\left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|.$$

Thus, the volume of a tetrahedron spanned by three vectors is one-sixth of the volume of the parallelepiped spanned by the same vectors.

15. Find the volume of the tetrahedron with vertices $(1, 0, 0)$, $(1, 2, 0)$, $(2, 2, 2)$, and $(0, 3, 2)$.

16. Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

17. For what value of k do the four points $(1, 1, -1)$, $(0, 3, -2)$, $(-2, 1, 0)$, and $(k, 0, 2)$ all lie in a plane?

18. **(The scalar triple product)** Verify the identities

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}).$$

19. If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) \ne 0$ and $\mathbf{x}$ is an arbitrary 3-vector, find the numbers λ, μ, and ν such that

$$\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}.$$

20. If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 0$ but $\mathbf{v} \times \mathbf{w} \ne \mathbf{0}$, show that there are constants λ and μ such that

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w}.$$

Hint: Use the result of Exercise 19 with $\mathbf{u}$ in place of $\mathbf{x}$ and $\mathbf{v} \times \mathbf{w}$ in place of $\mathbf{u}$.

21. Calculate $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, given that $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$, and $\mathbf{w} = \mathbf{j} - \mathbf{k}$. Why would you not expect these to be equal?

22. Does the notation $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ make sense? Why? How about the notation $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$?

23. **(The vector triple product)** The product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called a **vector triple product**. Since it is perpendicular to $\mathbf{v} \times \mathbf{w}$, it must lie in the plane of $\mathbf{v}$ and $\mathbf{w}$. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}.$$

Hint: This can be done by direct calculation of the components of both sides of the equation, but the job is much easier if you choose coordinate axes so that $\mathbf{v}$ lies along the x-axis and $\mathbf{w}$ lies in the xy-plane.

24. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular vectors, show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$. What is $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$ in this case?

25. Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

26. Find all vectors $\mathbf{x}$ that satisfy the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{x} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

27. Show that the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{x} = \mathbf{i} + 5\mathbf{j}$$

has no solutions for the unknown vector $\mathbf{x}$.

28. What condition must be satisfied by the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ to guarantee that the equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ has a solution for $\mathbf{x}$? Is the solution unique?

EXERCISES 10.4

1. A single equation involving the coordinates (x, y, z) need not always represent a two-dimensional "surface" in $\mathbb{R}^3$. For example, $x^2 + y^2 + z^2 = 0$ represents the single point $(0, 0, 0)$, which has dimension zero. Give examples of single equations in x, y, and z that represent

(a) a (one-dimensional) straight line,

(b) the whole of $\mathbb{R}^3$,

(c) no points at all (i.e., the empty set).

In Exercises 2–9, find equations of the planes satisfying the given conditions.

2. Passing through $(0, 2, -3)$ and normal to the vector $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

3. Passing through the origin and having normal $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

4. Passing through $(1, 2, 3)$ and parallel to the plane $3x + y - 2z = 15$

5. Passing through the three points $(1, 1, 0)$, $(2, 0, 2)$, and $(0, 3, 3)$

6. Passing through the three points $(-2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$

7. Passing through $(1, 1, 1)$ and $(2, 0, 3)$ and perpendicular to the plane $x + 2y - 3z = 0$

8. Passing through the line of intersection of the planes $2x + 3y - z = 0$ and $x - 4y + 2z = -5$, and passing through the point $(-2, 0, -1)$

9. Passing through the line $x + y = 2$, $y - z = 3$, and perpendicular to the plane $2x + 3y + 4z = 5$

10. Under what geometric condition will three distinct points in $\mathbb{R}^3$ not determine a unique plane passing through them? How can this condition be expressed algebraically in terms of the position vectors, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, of the three points?

11. Give a condition on the position vectors of four points that guarantees that the four points are *coplanar*, that is, all lie on one plane.

Describe geometrically the one-parameter families of planes in Exercises 12–14. (λ is a real parameter.)

12. $x + y + z = \lambda$.

! **13.** $x + \lambda y + \lambda z = \lambda$.

! **14.** $\lambda x + \sqrt{1 - \lambda^2}\, y = 1$.

In Exercises 15–19, find equations of the line specified in vector and scalar parametric forms and in standard form.

15. Through the point $(1, 2, 3)$ and parallel to $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

16. Through $(-1, 0, 1)$ and perpendicular to the plane $2x - y + 7z = 12$

17. Through the origin and parallel to the line of intersection of the planes $x + 2y - z = 2$ and $2x - y + 4z = 5$

18. Through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$

19. Through $(1, 2, -1)$ and making equal angles with the positive directions of the coordinate axes

In Exercises 20–22, find the equations of the given line in standard form.

20. $\mathbf{r} = (1 - 2t)\mathbf{i} + (4 + 3t)\mathbf{j} + (9 - 4t)\mathbf{k}$.

21. $\begin{cases} x = 4 - 5t \\ y = 3t \\ z = 7 \end{cases}$

22. $\begin{cases} x - 2y + 3z = 0 \\ 2x + 3y - 4z = 4 \end{cases}$

23. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, show that the equations

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$

represent a line through P_1 and P_2.

24. What points on the line in Exercise 23 correspond to the parameter values $t = -1$, $t = 1/2$, and $t = 2$? Describe their locations.

25. Under what conditions on the position vectors of four distinct points P_1, P_2, P_3, and P_4 will the straight line through P_1 and P_2 intersect the straight line through P_3 and P_4 at a unique point?

Find the required distances in Exercises 26–29.

26. From the origin to the plane $x + 2y + 3z = 4$

27. From $(1, 2, 0)$ to the plane $3x - 4y - 5z = 2$

28. From the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$

29. Between the lines

$$\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$$

30. Show that the line $x - 2 = \dfrac{y + 3}{2} = \dfrac{z - 1}{4}$ is parallel to the plane $2y - z = 1$. What is the distance between the line and the plane?

In Exercises 31–32, describe the one-parameter families of straight lines represented by the given equations. (λ is a real parameter.)

! **31.** $(1 - \lambda)(x - x_0) = \lambda(y - y_0)$, $z = z_0$.

! **32.** $\dfrac{x - x_0}{\sqrt{1 - \lambda^2}} = \dfrac{y - y_0}{\lambda} = z - z_0$.

33. Why does the factored second-degree equation

$$(A_1x + B_1y + C_1z - D_1)(A_2x + B_2y + C_2z - D_2) = 0$$

represent a pair of planes rather than a single straight line?

EXERCISES 10.5

Identify the surfaces represented by the equations in Exercises 1–16 and sketch their graphs.

1. $x^2 + 4y^2 + 9z^2 = 36$ **2.** $x^2 + y^2 + 4z^2 = 4$

3. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

4. $x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$

5. $z = x^2 + 2y^2$ **6.** $z = x^2 - 2y^2$

7. $x^2 - y^2 - z^2 = 4$ **8.** $-x^2 + y^2 + z^2 = 4$

9. $z = xy$ **10.** $x^2 + 4z^2 = 4$

11. $x^2 - 4z^2 = 4$ **12.** $y = z^2$

13. $x = z^2 + z$ **14.** $x^2 = y^2 + 2z^2$

15. $(z - 1)^2 = (x - 2)^2 + (y - 3)^2$

16. $(z - 1)^2 = (x - 2)^2 + (y - 3)^2 + 4$

Describe and sketch the geometric objects represented by the systems of equations in Exercises 17–20.

17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$ **18.** $\begin{cases} x^2 + y^2 = 1 \\ z = x + y \end{cases}$

19. $\begin{cases} z^2 = x^2 + y^2 \\ z = 1 + x \end{cases}$ **20.** $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$

21. Find two one-parameter families of straight lines that lie on the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

22. Find two one-parameter families of straight lines that lie on the hyperbolic paraboloid $z = xy$.

23. The equation $2x^2 + y^2 = 1$ represents a cylinder with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cylinder has circular cross-sections.

! **24.** The equation $z^2 = 2x^2 + y^2$ represents a cone with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cone has circular cross-sections. *Hint:* Do Exercise 23 first and use its result.

EXERCISES 10.6

1. Convert the Cartesian coordinates $(2, -2, 1)$ to cylindrical coordinates and to spherical coordinates.
2. Convert the cylindrical coordinates $[2, \pi/6, -2]$ to Cartesian coordinates and to spherical coordinates.
3. Convert the spherical coordinates $[4, \pi/3, 2\pi/3]$ to Cartesian coordinates and to cylindrical coordinates.
4. A point P has spherical coordinates $[1, \phi, \theta]$ and cylindrical coordinates $[r, \pi/4, r]$. Find the Cartesian coordinates of the point.

Describe the sets of points in 3-space that satisfy the equations in Exercises 5–14. Here, r, θ, R, and ϕ denote the appropriate cylindrical or spherical coordinates.

5. $\theta = \pi/2$
6. $\phi = 2\pi/3$
7. $\phi = \pi/2$
8. $R = 4$
9. $r = 4$
10. $R = z$
11. $R = r$
12. $R = 2x$
13. $R = 2\cos\phi$
14. $r = 2\cos\theta$

EXERCISES 11.1

In Exercises 1–14, find the velocity, speed, and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

1. $\mathbf{r} = \mathbf{i} + t\mathbf{j}$

2. $\mathbf{r} = t^2\mathbf{i} + \mathbf{k}$

3. $\mathbf{r} = t^2\mathbf{j} + t\mathbf{k}$

4. $\mathbf{r} = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$

5. $\mathbf{r} = t^2\mathbf{i} - t^2\mathbf{j} + \mathbf{k}$

6. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$

7. $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$

8. $\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\mathbf{j} + a\sin\omega t\,\mathbf{k}$

9. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 5\sin t\,\mathbf{k}$

10. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}$

11. $\mathbf{r} = ae^t\mathbf{i} + be^t\mathbf{j} + ce^t\mathbf{k}$

12. $\mathbf{r} = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$

13. $\mathbf{r} = e^{-t}\cos(e^t)\mathbf{i} + e^{-t}\sin(e^t)\mathbf{j} - e^t\mathbf{k}$

14. $\mathbf{r} = a\cos t\sin t\,\mathbf{i} + a\sin^2 t\,\mathbf{j} + a\cos t\,\mathbf{k}$

15. A particle moves around the circle $x^2 + y^2 = 25$ at constant speed, making one revolution in 2 s. Find its acceleration when it is at $(3, 4)$.

16. A particle moves to the right along the curve $y = 3/x$. If its speed is 10 when it passes through the point $\left(2, \frac{3}{2}\right)$, what is its velocity at that time?

17. A point P moves along the curve of intersection of the cylinder $z = x^2$ and the plane $x + y = 2$ in the direction of increasing y with constant speed $v = 3$. Find the velocity of P when it is at $(1, 1, 1)$.

18. An object moves along the curve $y = x^2$, $z = x^3$, with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point $(2, 4, 8)$.

19. A particle moves along the curve $\mathbf{r} = 3u\mathbf{i} + 3u^2\mathbf{j} + 2u^3\mathbf{k}$ in the direction corresponding to increasing u and with a constant speed of 6. Find the velocity and acceleration of the particle when it is at the point $(3, 3, 2)$.

20. A particle moves along the curve of intersection of the cylinders $y = -x^2$ and $z = x^2$ in the direction in which x increases. (All distances are in centimetres.) At the instant when the particle is at the point $(1, -1, 1)$, its speed is 9 cm/s, and that speed is increasing at a rate of 3 cm/s^2. Find the velocity and acceleration of the particle at that instant.

21. Show that if the dot product of the velocity and acceleration of a moving particle is positive (or negative), then the speed of the particle is increasing (or decreasing).

22. Verify the formula for the derivative of a dot product given in Theorem 1(c).

23. Verify the formula for the derivative of a 3×3 determinant in the second remark following Theorem 1. Use this formula to verify the formula for the derivative of the cross product in Theorem 1.

24. If the position and velocity vectors of a moving particle are always perpendicular, show that the path of the particle lies on a sphere.

25. Generalize Exercise 24 to the case where the velocity of the particle is always perpendicular to the line joining the particle to a fixed point P_0.

26. What can be said about the motion of a particle at a time when its position and velocity satisfy $\mathbf{r} \bullet \mathbf{v} > 0$? What can be said when $\mathbf{r} \bullet \mathbf{v} < 0$?

In Exercises 27–32, assume that the vector functions encountered have continuous derivatives of all required orders.

27. Show that $\dfrac{d}{dt}\left(\dfrac{d\mathbf{u}}{dt} \times \dfrac{d^2\mathbf{u}}{dt^2}\right) = \dfrac{d\mathbf{u}}{dt} \times \dfrac{d^3\mathbf{u}}{dt^3}$.

28. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\Big)$.

29. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\Big)$.

30. Expand and simplify: $\dfrac{d}{dt}\left(\mathbf{u} \times \left(\dfrac{d\mathbf{u}}{dt} \times \dfrac{d^2\mathbf{u}}{dt^2}\right)\right)$.

31. Expand and simplify: $\dfrac{d}{dt}\Big((\mathbf{u} + \mathbf{u}'') \bullet (\mathbf{u} \times \mathbf{u}')\Big)$.

32. Expand and simplify: $\dfrac{d}{dt}\Big((\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'')\Big)$.

33. If at all times t the position and velocity vectors of a moving particle satisfy $\mathbf{v}(t) = 2\mathbf{r}(t)$, and if $\mathbf{r}(0) = \mathbf{r}_0$, find $\mathbf{r}(t)$ and the acceleration $\mathbf{a}(t)$. What is the path of motion?

✻ **34.** Verify that $\mathbf{r} = \mathbf{r}_0\cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)$ satisfies the initial-value problem

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}, \qquad \mathbf{r}'(0) = \mathbf{v}_0, \qquad \mathbf{r}(0) = \mathbf{r}_0.$$

(It is the unique solution.) Describe the path $\mathbf{r}(t)$. What is the path if $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$?

✻ **35. (Free fall with air resistance)** A projectile falling under gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - c\frac{d\mathbf{r}}{dt},$$

where c is a positive constant. If $\mathbf{r} = \mathbf{r}_0$ and $d\mathbf{r}/dt = \mathbf{v}_0$ at time $t = 0$, find $\mathbf{r}(t)$. (*Hint:* Let $\mathbf{w} = e^{ct}(d\mathbf{r}/dt)$.) Show that the solution approaches that of the projectile problem given in this section as $c \to 0$.

EXERCISES 11.3

In Exercises 1–4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise

2. In terms of the x-coordinate, oriented clockwise

3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

4. In terms of arc length measured from $(0, a)$, oriented clockwise

5. The cylinders $z = x^2$ and $z = 4y^2$ intersect in two curves, one of which passes through the point $(2, -1, 4)$. Find a parametrization of that curve using $t = y$ as parameter.

6. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

In Exercises 7–10, parametrize the curve of intersection of the given surfaces. *Note*: the answers are not unique.

7. $x^2 + y^2 = 9$ and $z = x + y$

8. $z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$

9. $z = x^2 + y^2$ and $2x - 4y - z - 1 = 0$

10. $yz + x = 1$ and $xz - x = 1$

11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter: (a) $t = x$, (b) $t = y$, and (c) $t = z$. Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices?

! **12.** The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathcal{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathcal{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathcal{C}$. (*Hint:* To be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathcal{C}$.

13. Find the length of the curve $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.

14. For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le T)$ given by $s(T) = T + T^3$?

15. Express the length of the curve $\mathbf{r} = at^2\,\mathbf{i} + bt\,\mathbf{j} + c\ln t\,\mathbf{k}$, $(1 \le t \le T)$, as a definite integral. Evaluate the integral if $b^2 = 4ac$.

16. Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

17. Find the length of the conical helix given by the parametrization $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $(0 \le t \le 2\pi)$. Why is the curve called a conical helix?

18. Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

19. Let $\mathcal{C}$ be the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = t$ between $t = 0$ and $t = 2\pi$. Find the length of $\mathcal{C}$.

20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.

21. Describe the piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$, and $\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 + t)\mathbf{j}$, $(0 \le t \le 1)$.

! **22.** A cable of length L and circular cross-section of radius a is wound around a cylindrical spool of radius b with no overlapping and with the adjacent windings touching one another. What length of the spool is covered by the cable?

In Exercises 23–26, reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t = 0$.

23. $\mathbf{r} = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}, \qquad (A^2 + B^2 + C^2 > 0)$

24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$

! **25.** $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}, \qquad (0 \le t \le \frac{\pi}{2})$

! **26.** $\mathbf{r} = 3t\cos t\,\mathbf{i} + 3t\sin t\,\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}$

? **27.** Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \le t \le b)$, and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \le u \le d)$, be two parametrizations of the same curve $\mathcal{C}$, each one-to-one on its domain and each giving $\mathcal{C}$ the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$). Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left|\frac{d}{dt}\mathbf{r}_1(t)\right| dt = \int_c^d \left|\frac{d}{du}\mathbf{r}_2(u)\right| du,$$

and thus that the length of $\mathcal{C}$ is independent of parametrization.

? **28.** If the curve $\mathbf{r} = \mathbf{r}(t)$ has continuous, nonvanishing velocity $\mathbf{v}(t)$ on the interval $[a, b]$, and if t_0 is some point in $[a, b]$, show that the function

$$s = g(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\,du$$

is an increasing function on $[a, b]$ and so has an inverse:

$$t = g^{-1}(s) \iff s = g(t).$$

Hence, show that the curve can be parametrized in terms of arc length measured from $\mathbf{r}(t_0)$.

EXERCISES 12.3

In Exercises 1–10, find all the first partial derivatives of the function specified, and evaluate them at the given point.

1. $f(x, y) = x - y + 2, \quad (3, 2)$

2. $f(x, y) = xy + x^2, \quad (2, 0)$

3. $f(x, y, z) = x^3y^4z^5, \quad (0, -1, -1)$

4. $g(x, y, z) = \dfrac{xz}{y + z}, \quad (1, 1, 1)$

5. $z = \tan^{-1}\left(\dfrac{y}{x}\right), \quad (-1, 1)$

6. $w = \ln(1 + e^{xyz}), \quad (2, 0, -1)$

7. $f(x, y) = \sin(x\sqrt{y}), \quad \left(\dfrac{\pi}{3}, 4\right)$

8. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}, \quad (-3, 4)$

9. $w = x^{(y \ln z)}, \quad (e, 2, e)$

10. $g(x_1, x_2, x_3, x_4) = \dfrac{x_1 - x_2^2}{x_3 + x_4^2}, \quad (3, 1, -1, -2)$

In Exercises 11–12, calculate the first partial derivatives of the given functions at (0, 0). You will have to use Definition 4.

11. $f(x, y) = \begin{cases} \dfrac{2x^3 - y^3}{x^2 + 3y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

12. $f(x, y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

In Exercises 13–22, find equations of the tangent plane and normal line to the graph of the given function at the point with specified values of x and y.

13. $f(x, y) = x^2 - y^2$ at $(-2, 1)$

14. $f(x, y) = \dfrac{x - y}{x + y}$ at $(1, 1)$

15. $f(x, y) = \cos(x/y)$ at $(\pi, 4)$

16. $f(x, y) = e^{xy}$ at $(2, 0)$

17. $f(x, y) = \dfrac{x}{x^2 + y^2}$ at $(1, 2)$

18. $f(x, y) = y\,e^{-x^2}$ at $(0, 1)$

19. $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$

20. $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ at $(0, 2)$

21. $f(x, y) = \tan^{-1}(y/x)$ at $(1, -1)$

22. $f(x, y) = \sqrt{1 + x^3y^2}$ at $(2, 1)$

23. Find the coordinates of all points on the surface with equation $z = x^4 - 4xy^3 + 6y^2 - 2$ where the surface has a horizontal tangent plane.

24. Find all horizontal planes that are tangent to the surface with equation $z = xye^{-(x^2+y^2)/2}$. At what points are they tangent?

In Exercises 25–31, show that the given function satisfies the given partial differential equation.

✻**25.** $z = x\,e^y, \quad x\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$

✻**26.** $z = \dfrac{x + y}{x - y}, \quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$

✻**27.** $z = \sqrt{x^2 + y^2}, \quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

✻**28.** $w = x^2 + yz, \quad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 2w$

✻**29.** $w = \dfrac{1}{x^2 + y^2 + z^2}, \quad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = -2w$

✻**30.** $z = f(x^2 + y^2)$, where f is any differentiable function of one variable,

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

✻**31.** $z = f(x^2 - y^2)$, where f is any differentiable function of one variable,

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

32. Give a formal definition of the three first partial derivatives of the function $f(x, y, z)$.

33. What is an equation of the "tangent hyperplane" to the graph $w = f(x, y, z)$ at $\big(a, b, c, f(a, b, c)\big)$?

! 34. Find the distance from the point $(1, 1, 0)$ to the circular paraboloid with equation $z = x^2 + y^2$.

! 35. Find the distance from the point $(0, 0, 1)$ to the elliptic paraboloid having equation $z = x^2 + 2y^2$.

! 36. Let $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Note that f is not continuous at $(0, 0)$. (See Example 3 of Section 12.2.) Therefore, its graph is not smooth there. Show, however, that $f_1(0, 0)$ and $f_2(0, 0)$ both exist. Hence, the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

37. Determine $f_1(0, 0)$ and $f_2(0, 0)$ if they exist, where

$$f(x, y) = \begin{cases} (x^3 + y) \sin \dfrac{1}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

38. Calculate $f_1(x, y)$ for the function in Exercise 37. Is $f_1(x, y)$ continuous at $(0, 0)$?

! 39. Let $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Calculate $f_1(x, y)$ and $f_2(x, y)$ at all points (x, y) in the plane. Is f continuous at $(0, 0)$? Are f_1 and f_2 continuous at $(0, 0)$?

! 40. Let $f(x, y, z) = \begin{cases} \dfrac{xy^2z}{x^4 + y^4 + z^4}, & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0, & \text{if } (x, y, z) = (0, 0, 0). \end{cases}$

Find $f_1(0, 0, 0)$, $f_2(0, 0, 0)$, and $f_3(0, 0, 0)$. Is f continuous at $(0, 0, 0)$? Are f_1, f_2, and f_3 continuous at $(0, 0, 0)$?

EXERCISES 12.7

In Exercises 1–6, find:

(a) the gradient of the given function at the point indicated,

(b) an equation of the plane tangent to the graph of the given function at the point whose x and y coordinates are given, and

(c) an equation of the straight line tangent, at the given point, to the level curve of the given function passing through that point.

1. $f(x, y) = x^2 - y^2$ at $(2, -1)$

2. $f(x,y) = \dfrac{x-y}{x+y}$ at $(1,1)$

3. $f(x,y) = \dfrac{x}{x^2+y^2}$ at $(1,2)$

4. $f(x,y) = e^{xy}$ at $(2,0)$

5. $f(x,y) = \ln(x^2+y^2)$ at $(1,-2)$

6. $f(x,y) = \sqrt{1+xy^2}$ at $(2,-2)$

In Exercises 7–9, find an equation of the tangent plane to the level surface of the given function that passes through the given point.

7. $f(x,y,z) = x^2y + y^2z + z^2x$ at $(1,-1,1)$

8. $f(x,y,z) = \cos(x+2y+3z)$ at $\left(\dfrac{\pi}{2}, \pi, \pi\right)$

9. $f(x,y,z) = y\,e^{-x^2}\sin z$ at $(0,1,\pi/3)$

In Exercises 10–13, find the rate of change of the given function at the given point in the specified direction.

10. $f(x,y) = 3x - 4y$ at $(0,2)$ in the direction of the vector $-2\mathbf{i}$

11. $f(x,y) = x^2y$ at $(-1,-1)$ in the direction of the vector $\mathbf{i}+2\mathbf{j}$

12. $f(x,y) = \dfrac{x}{1+y}$ at $(0,0)$ in the direction of the vector $\mathbf{i}-\mathbf{j}$

13. $f(x,y) = x^2+y^2$ at $(1,-2)$ in the direction making a (positive) angle of 60° with the positive x-axis

14. Let $f(x,y) = \ln|\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Show that $\nabla f = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

15. Let $f(x,y,z) = |\mathbf{r}|^{-n}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Show that $\nabla f = \dfrac{-n\mathbf{r}}{|\mathbf{r}|^{n+2}}$.

16. Show that, in terms of polar coordinates (r,θ) (where $x = r\cos\theta$ and $y = r\sin\theta$), the gradient of a function $f(r,\theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\hat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\hat{\mathbf{r}}$ in the direction of increasing θ.

17. In what directions at the point $(2,0)$ does the function $f(x,y) = xy$ have rate of change -1? Are there directions in which the rate is -3? How about -2?

18. In what directions at the point (a,b,c) does the function $f(x,y,z) = x^2+y^2-z^2$ increase at half of its maximal rate at that point?

19. Find $\nabla f(a,b)$ for the differentiable function $f(x,y)$ given the directional derivatives

$$D_{(\mathbf{i}+\mathbf{j})/\sqrt{2}} f(a,b) = 3\sqrt{2} \text{ and } D_{(3\mathbf{i}-4\mathbf{j})/5} f(a,b) = 5.$$

20. If $f(x,y)$ is differentiable at (a,b), what condition should angles ϕ_1 and ϕ_2 satisfy in order that the gradient $\nabla f(a,b)$ can be determined from the values of the directional derivatives $D_{\phi_1} f(a,b)$ and $D_{\phi_2} f(a,b)$?

21. The temperature $T(x,y)$ at points of the xy-plane is given by $T(x,y) = x^2 - 2y^2$.

(a) Draw a contour diagram for T showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2,-1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moved from $(2,-1)$ at speed k in the direction of the vector $-\mathbf{i}-2\mathbf{j}$?

(e) Along what curve through $(2,-1)$ should the ant move in order to continue to experience maximum rate of cooling?

22. Find an equation of the curve in the xy-plane that passes through the point $(1,1)$ and intersects all level curves of the function $f(x,y) = x^4+y^2$ at right angles.

23. Find an equation of the curve in the xy-plane that passes through the point $(2,-1)$ and that intersects every curve with equation of the form $x^2y^3 = K$ at right angles.

24. Find the second directional derivative of $e^{-x^2-y^2}$ at the point $(a,b) \neq (0,0)$ in the direction directly away from the origin.

25. Find the second directional derivative of $f(x,y,z) = xyz$ at $(2,3,1)$ in the direction of the vector $\mathbf{i}-\mathbf{j}-\mathbf{k}$.

26. Find a vector tangent to the curve of intersection of the two cylinders $x^2+y^2 = 2$ and $y^2+z^2 = 2$ at the point $(1,-1,1)$.

27. Repeat Exercise 26 for the surfaces $x+y+z = 6$ and $x^2+y^2+z^2 = 14$ and the point $(1,2,3)$.

28. The temperature in 3-space is given by

$$T(x,y,z) = x^2 - y^2 + z^2 + xz^2.$$

At time $t = 0$ a fly passes through the point $(1,1,2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is 7, what rate of temperature change does it experience at $t = 0$?

29. State and prove a version of Theorem 6 for a function of three variables.

30. What is the level surface of $f(x,y,z) = \cos(x+2y+3z)$ that passes through (π,π,π)? What is the tangent plane to that level surface at that point? (Compare this exercise with Exercise 8 above.)

31. If $\nabla f(x,y) = 0$ throughout the disk $x^2+y^2 < r^2$, prove that $f(x,y)$ is constant throughout the disk.

32. Theorem 6 implies that the level curve of $f(x,y)$ passing through (a,b) is smooth (has a tangent line) at (a,b) provided f is differentiable at (a,b) and satisfies $\nabla f(a,b) \neq \mathbf{0}$. Show that the level curve need not be smooth at (a,b) if $\nabla f(a,b) = \mathbf{0}$. (*Hint:* Consider $f(x,y) = y^3 - x^2$ at $(0,0)$.)

33. If $\mathbf{v}$ is a nonzero vector, express $D_{\mathbf{v}}(D_{\mathbf{v}} f)$ in terms of the components of $\mathbf{v}$ and the second partials of f. What is the interpretation of this quantity for a moving observer?

34. An observer moves so that his position, velocity, and acceleration at time t are given by the formulas $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $\mathbf{v}(t) = d\mathbf{r}/dt$, and $\mathbf{a}(t) = d\mathbf{v}/dt$. If the temperature in the vicinity of the observer depends only on position, $T = T(x,y,z)$, express the second time derivative of temperature as measured by the observer in terms of $D_{\mathbf{v}}$ and $D_{\mathbf{a}}$.

35. Repeat Exercise 34 but with T depending explicitly on time as well as position: $T = T(x, y, z, t)$.

36. Let $f(x, y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Calculate $\nabla f(0, 0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}} f(0, 0)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

(c) Is $f(x, y)$ differentiable at $(0, 0)$? Why?

37. Let $f(x, y) = \begin{cases} 2x^2y/(x^4 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Use the definition of directional derivative as a limit (Definition 7) to show that $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ in the plane. Specifically, show that $D_{\mathbf{u}} f(0, 0) = 0$ if $v = 0$, and $D_{\mathbf{u}} f(0, 0) = 2u^2/v$ if $v \neq 0$. However, as was shown in Example 4 in Section 12.2, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$, so it is not continuous there. Even if a function has directional derivatives in all directions at a point, it may not be continuous at that point.

EXERCISES 14.1

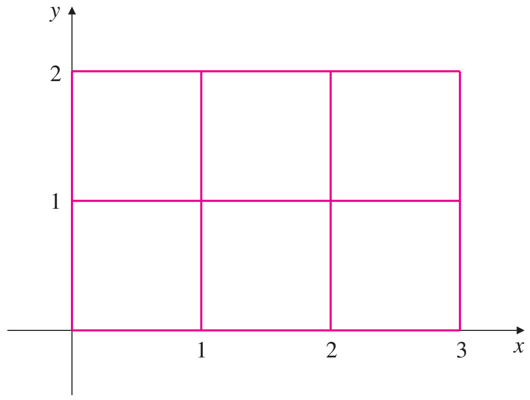

Figure 14.8

Exercises 1–6 refer to the double integral

$$I = \iint_D (5 - x - y)\, dA,$$

where D is the rectangle $0 \le x \le 3$, $0 \le y \le 2$. P is the partition of D into six squares of side 1 as shown in Figure 14.8. In Exercises 1–5, calculate the Riemann sums for I corresponding to the given choices of points (x_{ij}^*, y_{ij}^*).

1. (x_{ij}^*, y_{ij}^*) is the upper-left corner of each square.

2. (x_{ij}^*, y_{ij}^*) is the upper-right corner of each square.

3. (x_{ij}^*, y_{ij}^*) is the lower-left corner of each square.

4. (x_{ij}^*, y_{ij}^*) is the lower-right corner of each square.

5. (x_{ij}^*, y_{ij}^*) is the centre of each square.

6. Evaluate I by interpreting it as a volume.

In Exercises 7–10, D is the disk $x^2 + y^2 \le 25$, and P is the partition of the square $-5 \le x \le 5$, $-5 \le y \le 5$ into one hundred 1×1 squares, as shown in Figure 14.9. Approximate the double integral

$$J = \iint_D f(x, y)\, dA,$$

where $f(x, y) = 1$ by calculating the Riemann sums $R(f, P)$ corresponding to the indicated choice of points in the small squares. *Hint:* Using symmetry will make the job easier.

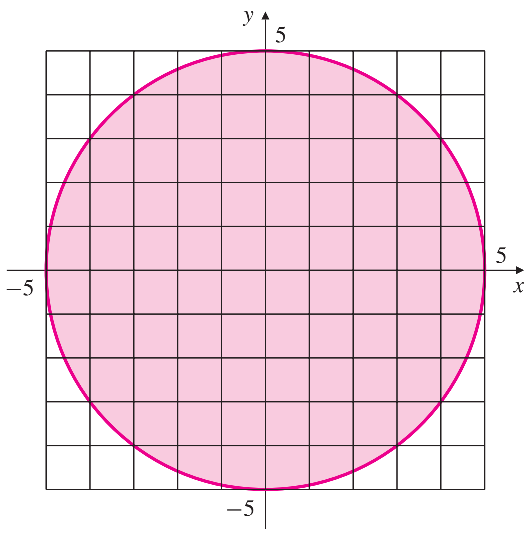

Figure 14.9

7. (x_{ij}^*, y_{ij}^*) is the corner of each square closest to the origin.

8. (x_{ij}^*, y_{ij}^*) is the corner of each square farthest from the origin.

9. (x_{ij}^*, y_{ij}^*) is the centre of each square.

10. Evaluate J.

11. Repeat Exercise 5 using the integrand e^x instead of $5 - x - y$.

12. Repeat Exercise 9 using $f(x, y) = x^2 + y^2$ instead of $f(x, y) = 1$.

In Exercises 13–22, evaluate the given double integral by inspection.

13. $\iint_R dA$, where R is the rectangle $-1 \le x \le 3$, $-4 \le y \le 1$

14. $\iint_D (x + 3)\, dA$, where D is the half-disk $0 \le y \le \sqrt{4 - x^2}$

15. $\iint_T (x + y)\, dA$, where T is the parallelogram having the points $(2, 2)$, $(1, -1)$, $(-2, -2)$, and $(-1, 1)$ as vertices

16. $\iint_{|x|+|y|\le 1} \left(x^3 \cos(y^2) + 3 \sin y - \pi\right) dA$

17. $\iint_{x^2+y^2\le 1} (4x^2y^3 - x + 5)\, dA$

18. $\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\, dA$

19. $\iint_{x^2+y^2\le a^2} (a - \sqrt{x^2 + y^2})\, dA$

20. $\iint_S (x + y)\, dA$, where S is the square $0 \le x \le a$, $0 \le y \le a$

21. $\iint_T (1 - x - y)\, dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

22. $\iint_R \sqrt{b^2 - y^2}\, dA$, where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

EXERCISES 14.2

In Exercises 1–4, calculate the given iterated integrals.

1. $\displaystyle\int_0^1 dx \int_0^x (xy + y^2)\, dy$ **2.** $\displaystyle\int_0^1 \int_0^y (xy + y^2)\, dx\, dy$

3. $\displaystyle\int_0^\pi \int_{-x}^x \cos y\, dy\, dx$ **4.** $\displaystyle\int_0^2 dy \int_0^y y^2 e^{xy}\, dx$

In Exercises 5–14, evaluate the double integrals by iteration.

5. $\displaystyle\iint_R (x^2 + y^2)\, dA$, where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

6. $\displaystyle\iint_R x^2 y^2\, dA$, where R is the rectangle of Exercise 5

7. $\displaystyle\iint_S (\sin x + \cos y)\, dA$, where S is the square $0 \le x \le \pi/2$, $0 \le y \le \pi/2$

8. $\displaystyle\iint_T (x - 3y)\, dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

9. $\displaystyle\iint_R xy^2\, dA$, where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

10. $\displaystyle\iint_D x \cos y\, dA$, where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$

11. $\displaystyle\iint_D \ln x\, dA$, where D is the finite region in the first quadrant bounded by the line $2x + 2y = 5$ and the hyperbola $xy = 1$

12. $\displaystyle\iint_T \sqrt{a^2 - y^2}\, dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, a)

13. $\displaystyle\iint_R \frac{x}{y} e^y\, dA$, where R is the region $0 \le x \le 1$, $x^2 \le y \le x$

14. $\displaystyle\iint_T \frac{xy}{1 + x^4}\, dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

In Exercises 15–18, sketch the domain of integration and evaluate the given iterated integrals.

15. $\displaystyle\int_0^1 dy \int_y^1 e^{-x^2}\, dx$ **16.** $\displaystyle\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\, dx$

17. $\displaystyle\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2 + y^2}\, dy \quad (\lambda > 0)$

18. $\displaystyle\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1 - y^4}\, dy$

In Exercises 19–28, find the volumes of the indicated solids.

19. Under $z = 1 - x^2$ and above the region $0 \le x \le 1$, $0 \le y \le x$

20. Under $z = 1 - x^2$ and above the region $0 \le y \le 1$, $0 \le x \le y$

21. Under $z = 1 - x^2 - y^2$ and above the region $x \ge 0$, $y \ge 0$, $x + y \le 1$

22. Under $z = 1 - y^2$ and above $z = x^2$

23. Under the surface $z = 1/(x + y)$ and above the region in the xy-plane bounded by $x = 1$, $x = 2$, $y = 0$, and $y = x$

24. Under the surface $z = x^2 \sin(y^4)$ and above the triangle in the xy-plane with vertices $(0, 0)$, $(0, \pi^{1/4})$, and $(\pi^{1/4}, \pi^{1/4})$

25. Above the xy-plane and under the surface $z = 1 - x^2 - 2y^2$

26. Above the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, and under the plane $z = 2 - (x/a) - (y/b)$

27. Inside the two cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$

28. Inside the cylinder $x^2 + 2y^2 = 8$, above the plane $z = y - 4$, and below the plane $z = 8 - x$

29. Suppose that $f(x, t)$ and $f_1(x, t)$ are continuous on the rectangle $a \le x \le b$ and $c \le t \le d$. Let

$$g(x) = \int_c^d f(x, t)\, dt \quad \text{and} \quad G(x) = \int_c^d f_1(x, t)\, dt.$$

Show that $g'(x) = G(x)$ for $a < x < b$. *Hint:* Evaluate $\int_a^x G(u)\, du$ by reversing the order of iteration. Then differentiate the result. This is a different version of Theorem 6 of Section 13.6.

30. Let $F'(x) = f(x)$ and $G'(x) = g(x)$ on the interval $a \le x \le b$. Let T be the triangle with vertices (a, a), (b, a), and (b, b). By iterating $\iint_T f(x)g(y)\, dA$ in both directions, show that

$$\int_a^b f(x)G(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b g(y)F(y)\, dy.$$

(This is an alternative derivation of the formula for integration by parts.)

31. Use Maple's **int** routine or similar routines in other computer algebra systems to evaluate the iterated integrals in Exercises 1–4 or the iterated integrals you constructed in the remaining exercises above.

EXERCISES 14.4

In Exercises 1–6, evaluate the given double integral over the disk D given by $x^2 + y^2 \le a^2$, where $a > 0$.

1. $\iint_D (x^2 + y^2)\, dA$

2. $\iint_D \sqrt{x^2 + y^2}\, dA$

3. $\iint_D \frac{1}{\sqrt{x^2 + y^2}}\, dA$

4. $\iint_D |x|\, dA$

5. $\iint_D x^2\, dA$

6. $\iint_D x^2 y^2\, dA$

In Exercises 7–10, evaluate the given double integral over the quarter-disk Q given by $x \ge 0$, $y \ge 0$, and $x^2 + y^2 \le a^2$, where $a > 0$.

7. $\iint_Q y\, dA$

8. $\iint_Q (x + y)\, dA$

9. $\iint_Q e^{x^2+y^2}\, dA$

10. $\iint_Q \frac{2xy}{x^2 + y^2}\, dA$

11. Evaluate $\iint_S (x + y)\, dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \le a^2$ and under the line $y = \sqrt{3}x$.

12. Find $\iint_S x\, dA$, where S is the disk segment $x^2 + y^2 \le 2$, $x \ge 1$.

13. Evaluate $\iint_T (x^2 + y^2)\, dA$, where T is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

14. Evaluate $\iint_{x^2+y^2 \le 1} \ln(x^2 + y^2)\, dA$.

15. Find the average distance from the origin to points in the disk $x^2 + y^2 \le a^2$.

16. Find the average value of $e^{-(x^2+y^2)}$ over the annular region $0 < a \le \sqrt{x^2 + y^2} \le b$.

17. For what values of k, and to what value, does the integral $\iint_{x^2+y^2 \le 1} \frac{dA}{(x^2 + y^2)^k}$ converge?

18. For what values of k, and to what value, does the integral $\iint_{\mathbb{R}^2} \frac{dA}{(1 + x^2 + y^2)^k}$ converge?

19. Evaluate $\iint_D xy\, dA$, where D is the plane region satisfying $x \ge 0$, $0 \le y \le x$, and $x^2 + y^2 \le a^2$.

20. Evaluate $\iint_C y\, dA$, where C is the upper half of the cardioid disk $r \le 1 + \cos\theta$.

21. Find the volume lying between the paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$.

22. Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

23. Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = 2a^2$ and the cylinder $x^2 + y^2 = a^2$.

24. Find the volume of the region lying above the xy-plane, inside the cylinder $x^2 + y^2 = 4$ and below the plane $z = x + y + 4$.

25. Find the volume of the region lying inside all three of the circular cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, and $y^2 + z^2 = a^2$. *Hint:* Make a good sketch of the first octant part of the region, and use symmetry whenever possible.

26. Find the volume of the region lying inside the circular cylinder $x^2 + y^2 = 2y$ and inside the parabolic cylinder $z^2 = y$.

27. Many points are chosen at random in the disk $x^2 + y^2 \le 1$. Find the approximate average value of the distance from these points to the nearest side of the smallest square that contains the disk.

28. Find the average value of x over the segment of the disk $x^2 + y^2 \le 4$ lying to the right of $x = 1$. What is the centroid of the segment?

29. Find the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

30. Find the volume of the region in the first octant below the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Hint: Use the change of variables $x = au$, $y = bv$.

31. Evaluate $\iint_{|x|+|y| \le a} e^{x+y}\, dA$.

32. Find $\iint_P (x^2 + y^2)\, dA$, where P is the parallelogram bounded by the lines $x + y = 1$, $x + y = 2$, $3x + 4y = 5$, and $3x + 4y = 6$.

33. Find the area of the region in the first quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$.

34. Evaluate $\iint_R (x^2 + y^2)\, dA$, where R is the region in the first quadrant bounded by $y = 0$, $y = x$, $xy = 1$, and $x^2 - y^2 = 1$.

35. Let T be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Evaluate the integral $\iint_T e^{(y-x)/(y+x)}\, dA$,

(a) by transforming to polar coordinates, and

(b) by using the transformation $u = y - x$, $v = y + x$.

36. Use the method of Example 7 to find the area of the region inside the ellipse $4x^2 + 9y^2 = 36$ and above the line $2x + 3y = 6$.

37. **(The error function)** The error function, $\operatorname{Erf}(x)$, is defined for $x \ge 0$ by

$$\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Show that $\left(\operatorname{Erf}(x)\right)^2 = \frac{4}{\pi} \int_0^{\pi/4} \left(1 - e^{-x^2/\cos^2\theta}\right) d\theta.$

Hence deduce that $\operatorname{Erf}(x) \ge \sqrt{1 - e^{-x^2}}$.

38. (The gamma and beta functions) The gamma function $\Gamma(x)$ and the beta function $B(x, y)$ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad (x > 0),$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad (x > 0,\ y > 0).$$

The gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x) \qquad \text{and}$$

$$\Gamma(n+1) = n!, \qquad (n = 0, 1, 2, \ldots).$$

Deduce the following further properties of these functions:

(a) $\Gamma(x) = 2\displaystyle\int_0^\infty s^{2x-1}e^{-s^2}\,ds, \qquad (x > 0),$

(b) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\dfrac{3}{2}\right) = \dfrac{1}{2}\sqrt{\pi},$

(c) If $x > 0$ and $y > 0$, then

$$B(x, y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta,$$

(d) $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$

EXERCISES 14.5

In Exercises 1–12, evaluate the triple integrals over the indicated region. Be alert for simplifications and auspicious orders of iteration.

1. $\iiint_R (1 + 2x - 3y)\, dV$, over the box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$

2. $\iiint_B xyz\, dV$, over the box B given by $0 \le x \le 1$, $-2 \le y \le 0$, $1 \le z \le 4$

3. $\iiint_D (3 + 2xy)\, dV$, over the solid hemispherical dome D given by $x^2 + y^2 + z^2 \le 4$ and $z \ge 0$

4. $\iiint_R x\, dV$, over the tetrahedron bounded by the coordinate planes and the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

5. $\iiint_R (x^2 + y^2)\, dV$, over the cube $0 \le x, y, z \le 1$

6. $\iiint_R (x^2 + y^2 + z^2)\, dV$, over the cube of Exercise 5

7. $\iiint_R (xy + z^2)\, dV$, over the set $0 \le z \le 1 - |x| - |y|$

8. $\iiint_R yz^2 e^{-xyz}\, dV$, over the cube $0 \le x, y, z \le 1$

9. $\iiint_R \sin(\pi y^3)\, dV$, over the pyramid with vertices (0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 1, 1), and (0, 1, 1)

10. $\iiint_R y\, dV$, over that part of the cube $0 \le x, y, z \le 1$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$

11. $\iiint_R \dfrac{1}{(x + y + z)^3}\, dV$, over the region bounded by the six planes $z = 1$, $z = 2$, $y = 0$, $y = z$, $x = 0$, and $x = y + z$

12. $\iiint_R \cos x \cos y \cos z\, dV$, over the tetrahedron defined by $x \ge 0$, $y \ge 0$, $z \ge 0$, and $x + y + z \le \pi$

13. Evaluate $\iiint_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2}\, dV$. *Hint:* Use the result of Example 4 of Section 14.4.

14. Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$, above the xy-plane, and below the plane $z = 2 + x$.

15. Find $\iiint_T x\, dV$, where T is the tetrahedron bounded by the planes $x = 1$, $y = 1$, $z = 1$, and $x + y + z = 2$.

16. Sketch the region R in the first octant of 3-space that has finite volume and is bounded by the surfaces $x = 0$, $z = 0$, $x + y = 1$, and $z = y^2$. Write six different iterations of the triple integral of $f(x, y, z)$ over R.

In Exercises 17–20, express the given iterated integral as a triple integral and sketch the region over which it is taken. Reiterate the integral, so that the outermost integral is with respect to x and the innermost is with respect to z.

17. $\displaystyle\int_0^1 dz \int_0^{1-z} dy \int_0^1 f(x, y, z)\, dx$

18. $\displaystyle\int_0^1 dz \int_z^1 dy \int_0^y f(x, y, z)\, dx$

19. $\displaystyle\int_0^1 dz \int_z^1 dx \int_0^{x-z} f(x, y, z)\, dy$

20. $\displaystyle\int_0^1 dy \int_0^{\sqrt{1-y^2}} dz \int_{y^2+z^2}^1 f(x, y, z)\, dx$

21. Repeat Exercise 17 using the method of Example 6.

22. Repeat Exercise 18 using the method of Example 6.

23. Repeat Exercise 19 using the method of Example 6.

24. Repeat Exercise 20 using the method of Example 6.

25. Rework Example 5 using the method of Example 6.

26. Rework Example 6 using the method of Example 5.

In Exercises 27–28, evaluate the given iterated integral by reiterating it in a different order. (You will need to make a good sketch of the region.)

27. $\displaystyle\int_0^1 dz \int_z^1 dx \int_0^x e^{x^3}\, dy$

28. $\displaystyle\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2 - z)}\, dz$

29. Define the average value of an integrable function $f(x, y, z)$ over a region R of 3-space. Find the average value of $x^2 + y^2 + z^2$ over the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

30. State a Mean-Value Theorem for triple integrals analogous to Theorem 3 of Section 14.3. Use it to prove that if $f(x, y, z)$ is continuous near the point (a, b, c) and if $B_\epsilon(a, b, c)$ is the ball of radius ϵ centred at (a, b, c), then

$$\lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \iiint_{B_\epsilon(a,b,c)} f(x, y, z)\, dV = f(a, b, c).$$

EXERCISES 14.6

In Exercises 1–9, find the volumes of the indicated regions.

1. Inside the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = a^2$
2. Above the surface $z = (x^2 + y^2)^{1/4}$ and inside the sphere $x^2 + y^2 + z^2 = 2$
3. Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$
4. Inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 12$
5. Above the xy-plane, inside the cone $z = 2a - \sqrt{x^2 + y^2}$, and inside the cylinder $x^2 + y^2 = 2ay$
6. Above the xy-plane, under the paraboloid $z = 1 - x^2 - y^2$, and in the wedge $-x \le y \le \sqrt{3}x$
7. In the first octant, between the planes $y = 0$ and $y = x$, and inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. *Hint:* Use the change of variables suggested in Example 1.
8. Bounded by the hyperboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ and the planes $z = -c$ and $z = c$
9. Above the xy-plane and below the paraboloid
$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$
10. Evaluate $\iiint_R (x^2 + y^2 + z^2)\,dV$, where R is the cylinder $0 \le x^2 + y^2 \le a^2$, $0 \le z \le h$.
11. Find $\iiint_B (x^2 + y^2)\,dV$, where B is the ball given by $x^2 + y^2 + z^2 \le a^2$.
12. Find $\iiint_B (x^2 + y^2 + z^2)\,dV$, where B is the ball of Exercise 11.
13. Find $\iiint_R (x^2 + y^2 + z^2)\,dV$, where R is the region that lies above the cone $z = c\sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = a^2$.
14. Evaluate $\iiint_R (x^2 + y^2)\,dV$ over the region R of Exercise 13.
15. Find $\iiint_R z\,dV$, over the region R satisfying $x^2 + y^2 \le z \le \sqrt{2 - x^2 - y^2}$.
16. Find $\iiint_R x\,dV$ and $\iiint_R z\,dV$, over that part of the hemisphere $0 \le z \le \sqrt{a^2 - x^2 - y^2}$ that lies in the first octant.
17. ! Find $\iiint_R x\,dV$ and $\iiint_R z\,dV$ over that part of the cone
$$0 \le z \le h\left(1 - \frac{\sqrt{x^2 + y^2}}{a}\right)$$
that lies in the first octant.
18. ! Find the volume of the region inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and above the plane $z = b - y$.
19. Show that for cylindrical coordinates the Laplace equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$
is given by
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$
20. ! Show that in spherical coordinates the Laplace equation is given by
$$\frac{\partial^2 u}{\partial R^2} + \frac{2}{R}\frac{\partial u}{\partial R} + \frac{\cot\phi}{R^2}\frac{\partial u}{\partial \phi} + \frac{1}{R^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$
21. ! If x, y, and z are functions of u, v, and w with continuous first partial derivatives and nonvanishing Jacobian at (u, v, w), show that they map an infinitesimal volume element in uvw-space bounded by the coordinate planes u, $u + du$, v, $v + dv$, w, and $w + dw$ into an infinitesimal "parallelepiped" in xyz-space having volume
$$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\,dv\,dw.$$
Hint: Adapt the two-dimensional argument given in Section 14.4. What three vectors from the point $P = (x(u, v, w), y(u, v, w), z(u, v, w))$ span the parallelepiped?

EXERCISES 15.1

In Exercises 1–8, sketch the given plane vector field and determine its field lines.

1. $\mathbf{F}(x, y) = x\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
3. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$
4. $\mathbf{F}(x, y) = \mathbf{i} + \sin x\,\mathbf{j}$
5. $\mathbf{F}(x, y) = e^x\mathbf{i} + e^{-x}\mathbf{j}$
6. $\mathbf{F}(x, y) = \nabla(x^2 - y)$
7. $\mathbf{F}(x, y) = \nabla \ln(x^2 + y^2)$
8. $\mathbf{F}(x, y) = \cos y\,\mathbf{i} - \cos x\,\mathbf{j}$

In Exercises 9–16, describe the streamlines of the given velocity fields.

9. $\mathbf{v}(x, y, z) = y\mathbf{i} - y\mathbf{j} - y\mathbf{k}$
10. $\mathbf{v}(x, y, z) = x\mathbf{i} + y\mathbf{j} - x\mathbf{k}$
11. $\mathbf{v}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$
12. $\mathbf{v}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j}}{(1 + z^2)(x^2 + y^2)}$
13. $\mathbf{v}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + x\mathbf{k}$
14. $\mathbf{v}(x, y, z) = e^{xyz}(x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k})$
15. $\mathbf{v}(x, y) = x^2\mathbf{i} - y\mathbf{j}$
16. $\mathbf{v}(x, y) = x\mathbf{i} + (x + y)\mathbf{j}$ *Hint:* Let $y = xv(x)$.

In Exercises 17–20, determine the field lines of the given polar vector fields.

17. $\mathbf{F} = \hat{\mathbf{r}} + r\hat{\boldsymbol{\theta}}$
18. $\mathbf{F} = \hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$
19. $\mathbf{F} = 2\hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$
20. $\mathbf{F} = r\hat{\mathbf{r}} - \hat{\boldsymbol{\theta}}$
21. Consider the Van der Pol equation with $\mu = 1$, so the corresponding vector field is $\mathbf{F} = y\mathbf{i} + \big(-x + y(1 - x^2)\big)\mathbf{j}$. Use $V(x, y) = x^2 - xy + y^2$ as in Example 7 to determine the stability of the the fixed point (0, 0).

22. Consider the vector field of the Van der Pol equation when $\mu = 0$. Use the Liapunov function, $V(x, y) = x^2 + y^2$, to attempt to determine the stability of the fixed point (0,0). Explain the result.
23. In Example 7, using the simpler Liapunov function, $V(x, y) = x^2 + y^2$, we found $V' = 2y^2(x^2 - 1) \le 0$. This was not sufficient to establish asymptotic stability in itself because $V' = 0$ occurs when $y = 0$. Zeros of V' form a curve, in this case given by the entire x axis, which all occur when $x' = 0$. Curves defined by one component of the vector field vanishing are known as **nulclines**. The zeros of V' occur on one nulcline (i.e., $y = 0$). Write an expression for another nulcline of the Van der Pol vector field of Example 7.
24. Give an alternative solution to Example 7 by using the fact that the simpler Liapunov function in the previous exercise is given by $V = r^2$ in polar coordinates. Show explicitly that all trajectories of the Van der Pol field (for $\mu = -1$) crossing the x axis stop moving toward (0, 0), by showing that $r(t)$ has a critical point. Then classify the associated critical point of $r(t)$ to demonstrate asymptotic stability.

EXERCISES 15.2

In Exercises 1–6, determine whether the given vector field is conservative, and find a potential if it is.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$

2. $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$

3. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} - y\mathbf{j}}{x^2 + y^2}$ **4.** $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$

5. $\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$

6. $\mathbf{F}(x, y, z) = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$

7. Find the three-dimensional vector field with potential $\phi(\mathbf{r}) = \dfrac{1}{|\mathbf{r} - \mathbf{r}_0|^2}$.

8. Calculate $\nabla \ln |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

! **9.** Show that the vector field

$$\mathbf{F}(x, y, z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} - \frac{x^2 + y^2}{z^2}\mathbf{k}$$

is conservative, and find its potential. Describe the equipotential surfaces. Find the field lines of $\mathbf{F}$.

! **10.** Repeat Exercise 9 for the field

$$\mathbf{F}(x, y, z) = \frac{2x}{z}\mathbf{i} + \frac{2y}{z}\mathbf{j} + \left(1 - \frac{x^2 + y^2}{z^2}\right)\mathbf{k}.$$

! **11.** Find the velocity field due to two sources of strength m, one located at $(0, 0, \ell)$ and the other at $(0, 0, -\ell)$. Where is the velocity zero? Find the velocity at any point $(x, y, 0)$ in the xy-plane. Where in the xy-plane is the speed greatest?

! **12.** Find the velocity field for a system consisting of a source of strength 2 at the origin and a sink of strength 1 at $(0, 0, 1)$. Show that the velocity is vertical at all points of a certain sphere. Sketch the streamlines of the flow.

Exercises 13–18 provide an analysis of two-dimensional sources and dipoles similar to that developed for three dimensions in the text.

13. In 3-space filled with an incompressible fluid, we say that the z-axis is a **line source** of strength m if every interval Δz along that axis emits fluid at volume rate $dV/dt = 2\pi m \Delta z$. The fluid then spreads out symmetrically in all directions perpendicular to the z-axis. Show that the velocity field of the flow is

$$\mathbf{v} = \frac{m}{x^2 + y^2}(x\mathbf{i} + y\mathbf{j}).$$

14. The flow in Exercise 13 is two-dimensional because $\mathbf{v}$ depends only on x and y and has no component in the z direction. Regarded as a *plane* vector field, it is the field of a two-dimensional point source of strength m located at the origin (i.e., fluid is emitted at the origin at the *areal rate* $dA/dt = 2\pi m$). Show that the vector field is conservative, and find a potential function $\phi(x, y)$ for it.

! **15.** Find the potential, ϕ, and the field, $\mathbf{F} = \nabla\phi$, for a two-dimensional dipole at the origin, with axis in the y direction and dipole moment μ. Such a dipole is the limit of a system consisting of a source of strength m at $(0, \ell/2)$ and a sink of strength m at $(0, -\ell/2)$, as $\ell \to 0$ and $m \to \infty$ such that $m\ell = \mu$.

16. Show that the equipotential curves of the two-dimensional dipole in Exercise 15 are circles tangent to the x-axis at the origin.

! **17.** Show that the streamlines (field lines) of the two-dimensional dipole in Exercises 15 and 16 are circles tangent to the y-axis at the origin. *Hint:* It is possible to do this geometrically. If you choose to do it by setting up a differential equation, you may find the change of dependent variable

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

useful for integrating the equation.

! **18.** Show that the velocity field of a line source of strength $2m$ can be found by integrating the (three-dimensional) velocity field of a point source of strength $m\,dz$ at $(0, 0, z)$ over the whole z-axis. Why does the integral correspond to a line source of strength $2m$ rather than strength m? Can the potential of the line source be obtained by integrating the potentials of the point sources?

19. Show that the gradient of a function expressed in terms of polar coordinates in the plane is

$$\nabla\phi(r, \theta) = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}.$$

(This is a repeat of Exercise 16 in Section 12.7.)

20. Use the result of Exercise 19 to show that a necessary condition for the vector field

$$\mathbf{F}(r, \theta) = F_r(r, \theta)\hat{\mathbf{r}} + F_\theta(r, \theta)\hat{\boldsymbol{\theta}}$$

(expressed in terms of polar coordinates) to be conservative is that

$$\frac{\partial F_r}{\partial\theta} - r\frac{\partial F_\theta}{\partial r} = F_\theta.$$

21. Show that $\mathbf{F} = r\sin 2\theta\,\hat{\mathbf{r}} + r\cos 2\theta\,\hat{\boldsymbol{\theta}}$ is conservative, and find a potential for it.

22. For what values of the constants α and β is the vector field

$$\mathbf{F} = r^2\cos\theta\,\hat{\mathbf{r}} + \alpha r^\beta \sin\theta\,\hat{\boldsymbol{\theta}}$$

conservative? Find a potential for $\mathbf{F}$ if α and β have these values.

EXERCISES 15.3

In Exercises 1–2 evaluate the given line integral over the specified curve $\mathcal{C}$.

1. $\int_{\mathcal{C}} (x+y)\,ds, \quad \mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k},\ 0 \le t \le m.$

2. $\int_{\mathcal{C}} y\,ds, \quad \mathbf{r} = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k},\ 0 \le t \le m.$

3. Show that the curve $\mathcal{C}$ given by

$$\mathbf{r} = a\cos t\sin t\,\mathbf{i} + a\sin^2 t\,\mathbf{j} + a\cos t\,\mathbf{k}, \quad (0 \le t \le \tfrac{\pi}{2}),$$

lies on a sphere centred at the origin. Find $\int_{\mathcal{C}} z\,ds$.

4. Let $\mathcal{C}$ be the conical helix with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$, $(0 \le t \le 2\pi)$. Find $\int_{\mathcal{C}} z\,ds$.

5. Find the mass of a wire along the curve

$$\mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}, \quad (0 \le t \le 1),$$

if the density at $\mathbf{r}(t)$ is $1 + t$ g/unit length.

6. Show that the curve $\mathcal{C}$ in Example 4 also has parametrization $x = \cos t$, $y = \sin t$, $z = \cos^2 t$, $(0 \le t \le \pi/2)$, and recalculate the mass of the wire in that example using this parametrization.

7. Find the moment of inertia about the z-axis (i.e., the value of $\delta \int_{\mathcal{C}} (x^2 + y^2)\,ds$), for a wire of constant density δ lying along the curve $\mathcal{C}$: $\mathbf{r} = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + t\mathbf{k}$, from $t = 0$ to $t = 2\pi$.

8. Evaluate $\int_{\mathcal{C}} e^z\,ds$, where $\mathcal{C}$ is the curve in Exercise 7.

9. Find $\int_{\mathcal{C}} x^2\,ds$ along the line of intersection of the two planes $x - y + z = 0$ and $x + y + 2z = 0$, from the origin to the point $(3, 1, -2)$.

10. Find $\int_{\mathcal{C}} \sqrt{1 + 4x^2z^2}\,ds$, where $\mathcal{C}$ is the curve of intersection of the surfaces $x^2 + z^2 = 1$ and $y = x^2$.

11. Find the mass and centre of mass of a wire bent in the shape of the circular helix $x = \cos t$, $y = \sin t$, $z = t$, $(0 \le t \le 2\pi)$, if the wire has line density given by $\delta(x, y, z) = z$.

12. Repeat Exercise 11 for the part of the wire corresponding to $0 \le t \le \pi$.

13. Find the moment of inertia about the y-axis of the curve $x = e^t$, $y = \sqrt{2}\,t$, $z = e^{-t}$, $(0 \le t \le 1)$, that is,

$$\int_{\mathcal{C}} (x^2 + z^2)\,ds.$$

14. Find the centroid of the curve in Exercise 13.

! 15. Find $\int_{\mathcal{C}} x\,ds$ along the first octant part of the curve of intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $z = x$.

! 16. Find $\int_{\mathcal{C}} z\,ds$ along the part of the curve $x^2 + y^2 + z^2 = 1$, $x + y = 1$, where $z \ge 0$.

! 17. Find $\int_{\mathcal{C}} \frac{ds}{(2y^2+1)^{3/2}}$, where $\mathcal{C}$ is the parabola $z^2 = x^2 + y^2$, $x + z = 1$. *Hint:* Use $y = t$ as parameter.

18. Express as a definite integral, but do not try to evaluate, the value of $\int_{\mathcal{C}} xyz\,ds$, where $\mathcal{C}$ is the curve $y = x^2$, $z = y^2$ from $(0, 0, 0)$ to $(2, 4, 16)$.

! 19. The function

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2\sin^2 t}\,dt$$

is called the **elliptic integral function of the second kind**. The **complete elliptic integral** of the second kind is the function $E(k) = E(k, \pi/2)$. In terms of these functions, express the length of one complete revolution of the elliptic helix

$$x = a\cos t, \qquad y = b\sin t, \qquad z = ct,$$

where $0 < a < b$. What is the length of that part of the helix lying between $t = 0$ and $t = T$, where $0 < T < \pi/2$?

! 20. Evaluate $\int_L \frac{ds}{x^2 + y^2}$, where L is the entire straight line with equation $Ax + By = C$, $(C \ne 0)$. *Hint:* Use the symmetry of the integrand to replace the line with a line having a simpler equation but giving the same value to the integral.

EXERCISES 15.4

In Exercises 1–6, evaluate the line integral of the tangential component of the given vector field along the given curve.

1. $\mathbf{F}(x, y) = xy\mathbf{i} - x^2\mathbf{j}$ along $y = x^2$ from $(0, 0)$ to $(1, 1)$
2. $\mathbf{F}(x, y) = \cos x\,\mathbf{i} - y\mathbf{j}$ along $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$
3. $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$ along the straight line from $(0, 0, 0)$ to $(1, 1, 1)$
4. $\mathbf{F}(x, y, z) = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$ along the curve $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$
5. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ from $(-1, 0, 0)$ to $(1, 0, 0)$ along either direction of the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$
6. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} - (x + y)\mathbf{k}$ along the polygonal path from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$
7. Find the work done by the force field
$$\mathbf{F} = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (z - y)\mathbf{k}$$
in moving an object from $(1, 0, -1)$ to $(0, -2, 3)$ along any smooth curve.
8. Evaluate $\oint_{\mathcal{C}} x^2y^2\,dx + x^3y\,dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.
9. Evaluate
$$\int_{\mathcal{C}} e^{x+y}\sin(y + z)\,dx + e^{x+y}\Big(\sin(y + z) + \cos(y + z)\Big)\,dy + e^{x+y}\cos(y + z)\,dz$$
along the straight line segment from (0,0,0) to $(1, \frac{\pi}{4}, \frac{\pi}{4})$.
10. The field $\mathbf{F} = (axy + z)\mathbf{i} + x^2\mathbf{j} + (bx + 2z)\mathbf{k}$ is conservative. Find a and b, and find a potential for $\mathbf{F}$. Also, evaluate $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where $\mathcal{C}$ is the curve from $(1, 1, 0)$ to $(0, 0, 3)$ that lies on the intersection of the surfaces $2x + y + z = 3$ and $9x^2 + 9y^2 + 2z^2 = 18$ in the octant $x \ge 0$, $y \ge 0$, $z \ge 0$.
11. Determine the values of A and B for which the vector field
$$\mathbf{F} = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k}$$
is conservative. If $\mathcal{C}$ is the straight line from $(1, 1, 1)$ to $(2, 1, 2)$, find
$$\int_{\mathcal{C}} 2x\ln z\,dx + 2y^2z\,dy + y^3\,dz.$$
12. Find the work done by the force field
$$\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$$
in moving a particle along the curve $x = \sin^{-1}t$, $y = 1 - 2t$, $z = 3t - 1$, $(0 \le t \le 1)$.
13. If $\mathcal{C}$ is the intersection of $z = \ln(1 + x)$ and $y = x$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$, evaluate
$$\int_{\mathcal{C}} \Big(2x\sin(\pi y) - e^z\Big)\,dx + \Big(\pi x^2\cos(\pi y) - 3e^z\Big)\,dy - xe^z\,dz.$$
14. Is each of the following sets a domain? a connected domain? a simply connected domain?
   (a) the set of points (x, y) in the plane such that $x > 0$ and $y \ge 0$
   (b) the set of points (x, y) in the plane such that $x = 0$ and $y \ge 0$
   (c) the set of points (x, y) in the plane such that $x \ne 0$ and $y > 0$
   (d) the set of points (x, y, z) in 3-space such that $x^2 > 1$
   (e) the set of points (x, y, z) in 3-space such that $x^2 + y^2 > 1$
   (f) the set of points (x, y, z) in 3-space such that $x^2 + y^2 + z^2 > 1$

In Exercises 15–19, evaluate the closed line integrals

(a) $\displaystyle\oint_{\mathcal{C}} x\,dy$, (b) $\displaystyle\oint_{\mathcal{C}} y\,dx$

around the given curves, all oriented counterclockwise.

15. The circle $x^2 + y^2 = a^2$
16. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
17. The boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$
18. The boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$
19. The triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$
20. On the basis of your results for Exercises 15–19, guess the values of the closed line integrals

   (a) $\displaystyle\oint_{\mathcal{C}} x\,dy$, (b) $\displaystyle\oint_{\mathcal{C}} y\,dx$

   for any non–self-intersecting closed curve in the xy-plane. Prove your guess in the case that $\mathcal{C}$ bounds a region of the plane that is both x-simple and y-simple. (See Section 14.2.)
21. If f and g are scalar fields with continuous first partial derivatives in a connected domain D, show that
$$\int_{\mathcal{C}} f\nabla g \bullet d\mathbf{r} + \int_{\mathcal{C}} g\nabla f \bullet d\mathbf{r} = f(Q)g(Q) - f(P)g(P)$$
for any piecewise smooth curve in D from P to Q.
22. Evaluate
$$\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{-y\,dx + x\,dy}{x^2 + y^2}$$
   (a) counterclockwise around the circle $x^2 + y^2 = a^2$,
   (b) clockwise around the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$,
   (c) counterclockwise around the boundary of the region $1 \le x^2 + y^2 \le 4$, $y \ge 0$.

23. Review Example 5 in Section 15.2 in which it was shown that

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right),$$

for all $(x, y) \neq (0, 0)$. Why does this result, together with that of Exercise 22, not contradict the final assertion in the remark following Theorem 1?

24. **(Winding number)** Let $\mathcal{C}$ be a piecewise smooth curve in the xy-plane that does not pass through the origin. Let $\theta = \theta(x, y)$ be the polar angle coordinate of the point $P = (x, y)$ on $\mathcal{C}$, not restricted to an interval of length 2π, but varying continuously as P moves from one end of $\mathcal{C}$ to the other. As in Example 5 of Section 15.2, it happens that

$$\nabla\theta = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}.$$

If, in addition, $\mathcal{C}$ is a closed curve, show that

$$w(\mathcal{C}) = \frac{1}{2\pi}\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2}$$

has an integer value. w is called the **winding number** of $\mathcal{C}$ about the origin.

EXERCISES 15.5

1. Verify that on the curve with polar equation $r = g(\theta)$ the arc length element is given by

$$ds = \sqrt{(g(\theta))^2 + (g'(\theta))^2}\, d\theta.$$

What is the area element on the vertical cylinder given in terms of cylindrical coordinates by $r = g(\theta)$?

2. Verify that on the spherical surface $x^2 + y^2 + z^2 = a^2$ the area element is given in terms of spherical coordinates by $dS = a^2 \sin\phi\, d\phi\, d\theta$.

3. Find the area of the part of the plane $Ax + By + Cz = D$ lying inside the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

5. State formulas for the surface area element dS for the surface with equation $F(x, y, z) = 0$ valid for the case where the surface has a one-to-one projection on (a) the xz-plane and (b) the yz-plane.

6. Repeat the area calculation of Example 8 by projecting the part of the surface shown in Figure 15.23 onto the yz-plane and using the formula in Exercise 5(b).

7. Find $\iint_{\mathcal{S}} x\, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

8. Find the area of the part of the cone $z^2 = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 2ay$.

9. Find the area of the part of the cylinder $x^2 + y^2 = 2ay$ that lies outside the cone $z^2 = x^2 + y^2$.

10. Find the area of the part of the cylinder $x^2 + z^2 = a^2$ that lies inside the cylinder $y^2 + z^2 = a^2$.

11. A circular cylinder of radius a is circumscribed about a sphere of radius a so that the cylinder is tangent to the sphere along the equator. Two planes, each perpendicular to the axis of the cylinder, intersect the sphere and the cylinder in circles. Show that the area of that part of the sphere between the two planes is equal to the area of the part of the cylinder between the two planes. Thus, the area of the part of a sphere between two parallel planes that intersect it depends only on the radius of the sphere and the distance between the planes, and not on the particular position of the planes.

12. Let $0 < a < b$. In terms of the elliptic integral functions defined in Exercise 19 of Section 15.3, find the area of that part of each of the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = b^2$ that lies inside the other cylinder.

13. Find $\iint_{\mathcal{S}} y\, dS$, where $\mathcal{S}$ is the part of the plane $z = 1 + y$ that lies inside the cone $z = \sqrt{2(x^2 + y^2)}$.

14. Find $\iint_{\mathcal{S}} y\, dS$, where $\mathcal{S}$ is the part of the cone $z = \sqrt{2(x^2 + y^2)}$ that lies below the plane $z = 1 + y$.

15. Find $\iint_{\mathcal{S}} xz\, dS$, where $\mathcal{S}$ is the part of the surface $z = x^2$ that lies in the first octant of 3-space and inside the paraboloid $z = 1 - 3x^2 - y^2$.

16. Find the mass of the part of the surface $z = \sqrt{2xy}$ that lies above the region $0 \le x \le 5$, $0 \le y \le 2$, if the areal density of the surface is $\sigma(x, y, z) = kz$.

17. Find the total charge on the surface

$$\mathbf{r} = e^u \cos v\,\mathbf{i} + e^u \sin v\,\mathbf{j} + u\mathbf{k}, \quad (0 \le u \le 1,\ 0 \le v \le \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Exercises 18–19 concern **spheroids**, which are ellipsoids with two of their three semi-axes equal, say $a = b$:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

18. Find the surface area of a **prolate spheroid**, where $0 < a < c$. A prolate spheroid has its two shorter semi-axes equal, like an American "pro football."

19. Find the surface area of an **oblate spheroid**, where $0 < c < a$. An oblate spheroid has its two longer semi-axes equal, like the earth.

20. Describe the parametric surface

$$x = au\cos v, \qquad y = au\sin v, \qquad z = bv,$$

$(0 \le u \le 1,\ 0 \le v \le 2\pi)$, and find its area.

21. Evaluate $\iint_{\mathcal{P}} \frac{dS}{(x^2 + y^2 + z^2)^{3/2}}$, where $\mathcal{P}$ is the plane with equation $Ax + By + Cz = D$, $(D \ne 0)$.

22. A spherical shell of radius a is centred at the origin. Find the centroid of that part of the sphere that lies in the first octant.

23. Find the centre of mass of a right-circular conical shell of base radius a, height h, and constant areal density σ.

24. Find the gravitational attraction of a hemispherical shell of radius a and constant areal density σ on a mass m located at the centre of the base of the hemisphere.

25. Find the gravitational attraction of a circular cylindrical shell of radius a, height h, and constant areal density σ on a mass m located on the axis of the cylinder b units above the base.

In Exercises 26–28, find the moment of inertia and radius of gyration of the given object about the given axis. Assume constant areal density σ in each case.

26. A cylindrical shell of radius a and height h about the axis of the cylinder

27. A spherical shell of radius a about a diameter

28. A right-circular conical shell of base radius a and height h about the axis of the cone

29. With what acceleration will the spherical shell of Exercise 27 roll down a plane inclined at angle α to the horizontal? (Compare your result with that of Example 4(b) of Section 14.7.)

EXERCISES 15.6

1. Find the flux of $\mathbf{F} = x\mathbf{i} + z\mathbf{j}$ out of the tetrahedron bounded by the coordinate planes and the plane $x + 2y + 3z = 6$.
2. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward across the sphere $x^2 + y^2 + z^2 = a^2$.
3. Find the flux of the vector field of Exercise 2 out of the surface of the box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.
4. Find the flux of the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{k}$ out across the boundary of the solid cone $0 \le z \le 1 - \sqrt{x^2 + y^2}$.
5. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the part of the surface $z = a - x^2 - y^2$ lying above plane $z = b < a$.
6. Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$.
7. Find the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.
8. Find the flux of $\mathbf{F} = z^2\mathbf{k}$ upward through the part of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant of 3-space.
9. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ upward through the part of the surface $z = 2 - x^2 - 2y^2$ that lies above the xy-plane.
10. Find the flux of $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the surface $\mathbf{r} = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$, $(0 \le u \le 1, 0 \le v \le 1)$.
11. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ upward through the surface $u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + u\, \mathbf{k}$, $(0 \le u \le 2, 0 \le v \le \pi)$.
12. Find the flux of $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + (x^2 + y^2)\mathbf{k}$ upward through the surface $\mathbf{r} = e^u \cos v\, \mathbf{i} + e^u \sin v\, \mathbf{j} + u\, \mathbf{k}$, where $0 \le u \le 1$ and $0 \le v \le \pi$.
13. Find the flux of $\mathbf{F} = m\mathbf{r}/|\mathbf{r}|^3$ out of the surface of the cube $-a \le x, y, z \le a$.
14. ! Find the flux of the vector field of Exercise 13 out of the box $1 \le x, y, z \le 2$. *Note:* This problem can be solved very easily using the Divergence Theorem of Section 16.4; the required flux is, in fact, zero. However, the object here is to do it by direct calculation of the surface integrals involved, and as such it is quite difficult. By symmetry, it is sufficient to evaluate the net flux out of the cube through any one of the three pairs of opposite faces; that is, you must calculate the flux through only two faces, say $z = 1$ and $z = 2$. Be prepared to work very hard to evaluate these integrals! When they are done, you may find the identities $2 \arctan a = \arctan\left(2a/(1 - a^2)\right)$ and $\arctan a + \arctan(1/a)\pi/2$ useful for showing that the net flux is zero.
15. Define the flux of a *plane* vector field across a piecewise smooth *curve*. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ outward across
    (a) the circle $x^2 + y^2 = a^2$, and
    (b) the boundary of the square $-1 \le x, y \le 1$.
16. Find the flux of $\mathbf{F} = -(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ *inward* across each of the two curves in the previous exercise.
17. If $\mathcal{S}$ is a smooth, oriented surface in 3-space and $\hat{\mathbf{N}}$ is the unit vector field determining the orientation of $\mathcal{S}$, show that the flux of $\hat{\mathbf{N}}$ across $\mathcal{S}$ is the area of $\mathcal{S}$.
18. ! The Divergence Theorem presented in Section 16.4 implies that the flux of a constant vector field across any oriented, piecewise smooth, closed surface is zero. Prove this now for (a) a rectangular box, and (b) a sphere.

CHAPTER REVIEW

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ vector field
  - ◇ scalar field
  - ◇ field line
  - ◇ conservative field
  - ◇ scalar potential
  - ◇ equipotential
  - ◇ a source
  - ◇ a dipole
  - ◇ connected domain
  - ◇ simply connected
  - ◇ parametric surface
  - ◇ orientable surface
  - ◇ the line integral of f along curve $\mathcal{C}$
  - ◇ the line integral of the tangential component of **F** along $\mathcal{C}$
  - ◇ the flux of a vector field through a surface
- **How are the field lines of a conservative field related to its equipotential curves or surfaces?**
- **How is a line integral of a scalar field calculated?**
- **How is a line integral of the tangential component of a vector field calculated?**
- **When is a line integral between two points independent of the path joining those points?**
- **How is a surface integral of a scalar field calculated?**
- **How do you calculate the flux of a vector field through a surface?**

Review Exercises

1. Find $\int_{\mathcal{C}} \frac{1}{y}\, ds$, where $\mathcal{C}$ is the curve

$$x = t, \quad y = 2e^t, \quad z = e^{2t}, \quad (-1 \le t \le 1).$$

2. Let $\mathcal{C}$ be the part of the curve of intersection of the surfaces $z = x + y^2$ and $y = 2x$ from the origin to the point $(2, 4, 18)$. Evaluate $\int_{\mathcal{C}} 2y\, dx + x\, dy + 2\, dz$.

3. Find $\iint_{\mathcal{S}} x\, dS$, where $\mathcal{S}$ is that part of the cone $z = \sqrt{x^2 + y^2}$ in the region $0 \le x \le 1 - y^2$.

4. Find $\iint_{\mathcal{S}} xyz\, dS$ over the part of the plane $x + y + z = 1$ lying in the first octant.

5. Find the flux of $x^2y\mathbf{i} - 10xy^2\mathbf{j}$ upward through the surface $z = xy$, $0 \le x \le 1$, $0 \le y \le 1$.

6. Find the flux of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ downward through the part of the plane $x + 2y + 3z = 6$ lying in the first octant.

7. A bead of mass m slides down a wire in the shape of the curve $x = a \sin t$, $\quad y = a \cos t$, $\quad z = bt$, where $0 \le t \le 6\pi$.

(a) What is the work done by the gravitational force $\mathbf{F} = -mg\mathbf{k}$ on the bead during its descent?

(b) What is the work done against a resistance of constant magnitude R which directly opposes the motion of the bead during its descent?

8. For what values of the constants a, b, and c can you determine the value of the integral I of the tangential component of $\mathbf{F} = (axy + 3yz)\mathbf{i} + (x^2 + 3xz + by^2z)\mathbf{j} + (bxy + cy^3)\mathbf{k}$ along a curve from $(0, 1, -1)$ to $(2, 1, 1)$ without knowing exactly which curve? What is the value of the integral?

9. Let $\mathbf{F} = (x^2/y)\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

(a) Find the field line of $\mathbf{F}$ that passes through $(1, 1, 0)$ and show that it also passes through $(e, e, 1)$.

(b) Find $\int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where $\mathcal{C}$ is the part of the field line in (a) from $(1, 1, 0)$ to $(e, e, 1)$.

10. Consider the vector fields

$$\mathbf{F} = (1 + x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2y)\mathbf{j} - 2z\mathbf{k},$$
$$\mathbf{G} = (1 + x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2z)\mathbf{j} - 2y\mathbf{k}.$$

(a) Show that $\mathbf{F}$ is conservative by finding a potential for it.

(b) Evaluate $\int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r}$, where $\mathcal{C}$ is given by

$$\mathbf{r} = (1 - t)e^t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \quad (0 \le t \le 1),$$

by taking advantage of the similarity between $\mathbf{F}$ and $\mathbf{G}$.

11. Find a plane vector field $\mathbf{F}(x, y)$ that satisfies the following conditions:

(i) The field lines of $\mathbf{F}$ are the curves $xy = C$.

(ii) $|\mathbf{F}(x, y)| = 1$ if $(x, y) \ne (0, 0)$.

(iii) $\mathbf{F}(1, 1) = (\mathbf{i} - \mathbf{j})/\sqrt{2}$.

(iv) $\mathbf{F}$ is continuous except at $(0, 0)$.

12. Let $\mathcal{S}$ be the part of the surface of the cylinder $y^2 + z^2 = 16$ that lies in the first octant and between the planes $x = 0$ and $x = 5$. Find the flux of $3z^2x\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ away from the x-axis through $\mathcal{S}$.

Challenging Problems

1. Find the centroid of the surface

$$\mathbf{r} = (2 + \cos v)(\cos u\mathbf{i} + \sin u\mathbf{j}) + \sin v\mathbf{k},$$

where $0 \le u \le 2\pi$ and $0 \le v \le \pi$. Describe this surface.

2. A smooth surface $\mathcal{S}$ is given parametrically by

$$\mathbf{r} = (\cos 2u)(2 + v\cos u)\mathbf{i} + (\sin 2u)(2 + v\cos u)\mathbf{j} + v\sin u\mathbf{k},$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Show that for *every* smooth vector field $\mathbf{F}$ on $\mathcal{S}$,

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 0,$$

where $\hat{\mathbf{N}} = \hat{\mathbf{N}}(u, v)$ is a unit normal vector field on $\mathcal{S}$ that depends continuously on (u, v). How do you explain this? *Hint:* Try to describe what the surface $\mathcal{S}$ looks like.

3. Recalculate the gravitational force exerted by a sphere of radius a and areal density σ centred at the origin on a point mass located at $(0, 0, b)$ by directly integrating the vertical component of the force due to an area element dS, rather than by integrating the potential as we did in the last part of Section 15.5. You will have to be quite creative in dealing with the resulting integral.

EXERCISES 16.1

In Exercises 1–11, calculate **div F** and **curl F** for the given vector fields.

1. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

2. $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$

3. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

4. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

5. $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$

6. $\mathbf{F} = xy^2\mathbf{i} - yz^2\mathbf{j} + zx^2\mathbf{k}$

7. $\mathbf{F} = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$

8. $\mathbf{F} = f(z)\mathbf{i} - f(z)\mathbf{j}$

9. $\mathbf{F}(r, \theta) = r\mathbf{i} + \sin\theta\mathbf{j}$, where (r, θ) are polar coordinates in the plane

10. $\mathbf{F} = \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

11. $\mathbf{F} = \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

! **12.** Let **F** be a smooth, three-dimensional vector field. If $B_{a,b,c}$ is the surface of the box $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$, with outward normal $\hat{\mathbf{N}}$, show that

$$\lim_{a,b,c\to 0^+} \frac{1}{8abc} \oiint_{B_{a,b,c}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \nabla \bullet \mathbf{F}(0, 0, 0).$$

! **13.** Let **F** be a smooth two-dimensional vector field. If $\mathcal{C}_\epsilon$ is the circle of radius ϵ centred at the origin, and $\hat{\mathbf{N}}$ is the unit outward normal to $\mathcal{C}_\epsilon$, show that

$$\lim_{\epsilon\to 0^+} \frac{1}{\pi\epsilon^2} \oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, ds = \mathbf{div}\,\mathbf{F}(0, 0).$$

! **14.** Prove Theorem 2 in the special case that $\mathcal{C}_\epsilon$ is the circle in the xy-plane with parametrization $x = \epsilon\cos\theta$, $\quad y = \epsilon\sin\theta$, $(0 \le \theta \le 2\pi)$. In this case $\hat{\mathbf{N}} = \mathbf{k}$. *Hint:* Expand $\mathbf{F}(x, y, z)$ in a vector Taylor series about the origin as in the proof of Theorem 1, and calculate the circulation of individual terms around $\mathcal{C}_\epsilon$.

EXERCISES 16.3

1. Evaluate $\oint_{\mathcal{C}} (\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy$, where $\mathcal{C}$ is the boundary of the half-disk $x^2 + y^2 \le a^2$, $y \ge 0$, oriented counterclockwise.

2. Evaluate $\oint_{\mathcal{C}} (x^2 - xy)\,dx + (xy - y^2)\,dy$ clockwise around the triangle with vertices (0, 0), (1, 1), and (2, 0).

3. Evaluate $\oint_{\mathcal{C}} \left(x\sin(y^2) - y^2\right) dx + \left(x^2 y\cos(y^2) + 3x\right) dy$, where $\mathcal{C}$ is the counterclockwise boundary of the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$, and $(0, 2)$.

4. Evaluate $\oint_{\mathcal{C}} x^2 y\,dx - xy^2\,dy$, where $\mathcal{C}$ is the clockwise boundary of the region $0 \le y \le \sqrt{9 - x^2}$.

5. Use a line integral to find the plane area enclosed by the curve $\mathbf{r} = a\cos^3 t\,\mathbf{i} + b\sin^3 t\,\mathbf{j}$, $(0 \le t \le 2\pi)$.

6. We deduced the two-dimensional Divergence Theorem from Green's Theorem. Reverse the argument and use the two-dimensional Divergence Theorem to prove Green's Theorem.

7. Sketch the plane curve $\mathcal{C}$: $\mathbf{r} = \sin t\,\mathbf{i} + \sin 2t\,\mathbf{j}$, $(0 \le t \le 2\pi)$. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} = ye^{x^2}\mathbf{i} + x^3 e^y\mathbf{j}$.

8. If $\mathcal{C}$ is the positively oriented boundary of a plane region R having area A and centroid $(\bar{x}, \bar{y})$, interpret geometrically the line integral $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where (a) $\mathbf{F} = x^2\mathbf{j}$, (b) $\mathbf{F} = xy\mathbf{i}$, and (c) $\mathbf{F} = y^2\mathbf{i} + 3xy\mathbf{j}$.

! 9. **(Average values of harmonic functions)** If $u(x, y)$ is harmonic in a domain containing a disk of radius r with boundary C_r, then the average value of u around the circle is the value of u at the centre. Prove this by showing that the derivative of the average value with respect to r is zero using the Divergence Theorem and the harmonicity of u, and the fact that the limit of the average value as $r \to 0$ is the value of u at the centre.

EXERCISES 16.4

In Exercises 1–4, use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$

2. $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$

3. $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$

4. $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$

In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x - 2)^2 + y^2 + (z - 3)^2 \le 9$

6. The solid ellipsoid $x^2 + y^2 + 4(z - 1)^2 \le 4$

7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$

8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$

9. Let A be the area of a region D forming part of the surface of a sphere of radius R centred at the origin, and let V be the volume of the solid cone C consisting of all points on line segments joining the centre of the sphere to points in D. Show that $V = \dfrac{1}{3}AR$ by applying the Divergence Theorem to $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

10. Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface $\mathcal{S}$ with vertices at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

11. A conical domain with vertex $(0, 0, b)$ and axis along the z-axis has as base a disk of radius a in the xy-plane. Find the flux of

$$\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$$

upward through the conical part of the surface of the domain.

12. Find the flux of $\mathbf{F} = (y + xz)\mathbf{i} + (y + yz)\mathbf{j} - (2x + z^2)\mathbf{k}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

13. Let D be the region $x^2 + y^2 + z^2 \le 4a^2$, $x^2 + y^2 \ge a^2$. The surface $\mathcal{S}$ of D consists of a cylindrical part, $\mathcal{S}_1$, and a spherical part, $\mathcal{S}_2$. Evaluate the flux of

$$\mathbf{F} = (x + yz)\mathbf{i} + (y - xz)\mathbf{j} + (z - e^x \sin y)\mathbf{k}$$

out of D through (a) the whole surface $\mathcal{S}$, (b) the surface $\mathcal{S}_1$, and (c) the surface $\mathcal{S}_2$.

14. Evaluate $\displaystyle\iint_{\mathcal{S}} (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS$, where $\mathcal{S}$ is that part of the cylinder $y^2 + z^2 = 1$ that lies in the first octant and between the planes $x = 0$ and $x = 1$.

15. A solid region R has volume V and centroid at the point $(\bar{x}, \bar{y}, \bar{z})$. Find the flux of

$$\mathbf{F} = (x^2 - x - 2y)\mathbf{i} + (2y^2 + 3y - z)\mathbf{j} - (z^2 - 4z + xy)\mathbf{k}$$

out of R through its surface.

16. The plane $x + y + z = 0$ divides the cube $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$ into two parts. Let the lower part (with one vertex at $(-1, -1, -1)$) be D. Sketch D. Note that it has seven faces, one of which is hexagonal. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of D through each of its faces.

17. Let $\mathbf{F} = (x^2 + y + 2 + z^2)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3 + x)\mathbf{k}$. Let $a > 0$, and let $\mathcal{S}$ be the part of the spherical surface $x^2 + y^2 + z^2 = 2az + 3a^2$ that is above the xy-plane. Find the flux of $\mathbf{F}$ outward across $\mathcal{S}$.

18. A pile of wet sand having total volume 5π covers the disk $x^2 + y^2 \le 1$, $z = 0$. The momentum of water vapour is given by $\mathbf{F} = \mathbf{grad}\,\phi + \mu\,\mathbf{curl}\,\mathbf{G}$, where $\phi = x^2 - y^2 + z^2$ is the water concentration, $\mathbf{G} = \frac{1}{3}(-y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k})$, and μ is a constant. Find the flux of $\mathbf{F}$ upward through the top surface of the sand pile.

In Exercises 19–29, D is a three-dimensional domain satisfying the conditions of the Divergence Theorem, and $\mathcal{S}$ is its surface. $\hat{\mathbf{N}}$ is the unit outward (from D) normal field on $\mathcal{S}$. The functions ϕ and ψ are smooth scalar fields on D. Also, $\partial\phi/\partial n$ denotes the first directional derivative of ϕ in the direction of $\hat{\mathbf{N}}$ at any point on $\mathcal{S}$:

$$\frac{\partial\phi}{\partial n} = \nabla\phi \bullet \hat{\mathbf{N}}.$$

19. Show that $\displaystyle\oiint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 0$, where $\mathbf{F}$ is an arbitrary smooth vector field.

20. Show that the volume V of D is given by

$$V = \frac{1}{3}\oiint_{\mathcal{S}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS.$$

21. If D has volume V, show that

$$\bar{\mathbf{r}} = \frac{1}{2V}\oiint_{\mathcal{S}} (x^2 + y^2 + z^2)\hat{\mathbf{N}}\, dS$$

is the position vector of the centre of gravity of D.

22. Show that $\displaystyle\oiint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}}\, dS = 0$.

23. If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi\,\mathbf{div}\,\mathbf{F}\, dV + \iiint_D \nabla\phi \bullet \mathbf{F}\, dV = \oiint_{\mathcal{S}} \phi\mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

Hint: Use Theorem 3(b) from Section 16.2.

Properties of the Laplacian operator

24. If $\nabla^2\phi = 0$ in D and $\phi(x, y, z) = 0$ on $\mathcal{S}$, show that $\phi(x, y, z) = 0$ in D. *Hint:* Let $\mathbf{F} = \nabla\phi$ in Exercise 23.

25. **(Uniqueness for the Dirichlet problem)** The Dirichlet problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that this problem can have at most one solution $u(x, y, z)$. *Hint:* Suppose there are two solutions, u and v, and apply Exercise 24 to their difference $\phi = u - v$.

26. **(The Neumann problem)** If $\nabla^2\phi = 0$ in D and $\partial\phi/\partial n = 0$ on $\mathcal{S}$, show that $\nabla\phi(x, y, z) = 0$ on D. The Neumann problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ \dfrac{\partial}{\partial n} u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that, if D is connected, then any two solutions of the Neumann problem must differ by a constant on D.

27. Verify that $\displaystyle\iiint_D \nabla^2\phi\, dV = \oiint_{\mathcal{S}} \frac{\partial\phi}{\partial n}\, dS.$

28. Verify that

$$\iiint_D \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oiint_{\mathcal{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS.$$

29. By applying the Divergence Theorem to $\mathbf{F} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector, show that

$$\iiint_D \nabla\phi\, dV = \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\, dS.$$

30. Let P_0 be a fixed point, and for each $\epsilon > 0$ let D_ϵ be a domain with boundary $\mathcal{S}_\epsilon$ satisfying the conditions of the Divergence Theorem. Suppose that the maximum distance from P_0 to points P in D_ϵ approaches zero as $\epsilon \to 0+$. If D_ϵ has volume $\text{vol}(D_\epsilon)$, show that

$$\lim_{\epsilon\to 0+} \frac{1}{\text{vol}(D_\epsilon)} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \textbf{div}\, \mathbf{F}(P_0).$$

This generalizes Theorem 1 of Section 16.1.

EXERCISES 16.5

1. Evaluate $\oint_{\mathcal{C}} xy\,dx + yz\,dy + zx\,dz$ around the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1), oriented clockwise as seen from the point (1, 1, 1).

2. Evaluate $\oint_{\mathcal{C}} y\,dx - x\,dy + z^2\,dz$ around the curve $\mathcal{C}$ of intersection of the cylinders $z = y^2$ and $x^2 + y^2 = 4$, oriented counterclockwise as seen from a point high on the z-axis.

3. Evaluate $\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$ with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

4. Evaluate $\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is the surface $x^2 + y^2 + 2(z-1)^2 = 6$, $z \geq 0$, $\hat{\mathbf{N}}$ is the unit outward (away from the origin) normal on $\mathcal{S}$, and

$$\mathbf{F} = (xz - y^3 \cos z)\mathbf{i} + x^3 e^z \mathbf{j} + xyz\, e^{x^2+y^2+z^2}\mathbf{k}.$$

5. Use Stokes's Theorem to show that

$$\oint_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \sqrt{3}\,\pi a^2,$$

where $\mathcal{C}$ is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$.

6. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ around the curve

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin 2t\,\mathbf{k}, \quad (0 \leq t \leq 2\pi),$$

where

$$\mathbf{F} = (e^x - y^3)\mathbf{i} + (e^y + x^3)\mathbf{j} + e^z\mathbf{k}.$$

Hint: Show that $\mathcal{C}$ lies on the surface $z = 2xy$.

7. Find the circulation of $\mathbf{F} = -y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ around the oriented boundary of the part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy-plane and having normal field pointing upward.

8. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where

$$\mathbf{F} = ye^x\mathbf{i} + (x^2 + e^x)\mathbf{j} + z^2e^z\mathbf{k},$$

and $\mathcal{C}$ is the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \cos t - \sin t)\mathbf{k}$$

for $0 \leq t \leq 2\pi$. *Hint:* Use Stokes's Theorem, observing that $\mathcal{C}$ lies in a certain plane and has a circle as its projection onto the xy-plane. The integral can also be evaluated by using the techniques of Section 15.4.

9. Let $\mathcal{C}_1$ be the straight line joining (−1, 0, 0) to (1, 0, 0), and let $\mathcal{C}_2$ be the semicircle $x^2 + y^2 = 1$, $z = 0$, $y \geq 0$. Let $\mathcal{S}$ be a smooth surface joining $\mathcal{C}_1$ to $\mathcal{C}_2$ having upward normal, and let

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z+1)\mathbf{k}.$$

Find the values of α and β for which $I = \iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$ is independent of the choice of $\mathcal{S}$, and find the value of I for these values of α and β.

10. Let $\mathcal{C}$ be the curve $(x-1)^2 + 4y^2 = 16$, $2x + y + z = 3$, oriented counterclockwise when viewed from high on the z-axis. Let

$$\mathbf{F} = (z^2 + y^2 + \sin x^2)\mathbf{i} + (2xy + z)\mathbf{j}) + (xz + 2yz)\mathbf{k}.$$

Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$.

11. If $\mathcal{C}$ is the oriented boundary of surface $\mathcal{S}$, and ϕ and ψ are arbitrary smooth scalar fields, show that

$$\oint_{\mathcal{C}} \phi \nabla\psi \bullet d\mathbf{r} = -\oint_{\mathcal{C}} \psi\nabla\phi \bullet d\mathbf{r} = \iint_{\mathcal{S}} (\nabla\phi \times \nabla\psi) \bullet \hat{\mathbf{N}}\,dS.$$

Is $\nabla\phi \times \nabla\psi$ solenoidal? Find a vector potential for it.

12. Let $\mathcal{C}$ be a piecewise smooth, simple closed plane curve in $\mathbb{R}^3$, which lies in a plane with unit normal $\hat{\mathbf{N}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and has orientation inherited from that of the plane. Show that the plane area enclosed by $\mathcal{C}$ is

$$\frac{1}{2}\oint_{\mathcal{C}} (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz.$$

13. Use Stokes's Theorem to prove Theorem 2 of Section 16.1.

EXERCISES 16.6

1. **(Archimedes' principle)** A solid occupying region R with surface $\mathcal{S}$ is immersed in a liquid of constant density ρ. The pressure at depth h in the liquid is $\rho g h$, so the pressure satisfies $\nabla p = \rho \mathbf{g}$, where $\mathbf{g}$ is the (vector) constant acceleration of gravity. Over each surface element dS on $\mathcal{S}$ the pressure of the fluid exerts a force $-p\hat{\mathbf{N}}\,dS$ on the solid.

(a) Show that the resultant "buoyancy force" on the solid is

$$\mathbf{B} = -\iiint_R \rho \mathbf{g}\, dV.$$

Thus, the buoyancy force has the same magnitude as, and opposite direction to, the weight of the liquid displaced by the solid. This is Archimedes' principle.

(b) Extend the above result to the case where the solid is only partly submerged in the fluid.

2. By breaking the vector $\mathbf{F}(\mathbf{G} \bullet \hat{\mathbf{N}})$ into its separate components and applying the Divergence Theorem to each separately, show that

$$\oiint_{\mathcal{S}} \mathbf{F}(\mathbf{G} \bullet \hat{\mathbf{N}})\, dS = \iiint_D \Big(\mathbf{F}\, \mathbf{div}\, \mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}\Big)\, dV,$$

where $\hat{\mathbf{N}}$ is the unit outward normal on the surface $\mathcal{S}$ of the domain D.

3. **(Gauss's Law)** Show that the flux of the electric field $\mathbf{E}$ outward through a closed surface $\mathcal{S}$ in 3-space is $1/\epsilon_0$ times the total charge enclosed by $\mathcal{S}$.

4. If $\mathbf{s} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$ and $f(\xi, \eta, \zeta)$ is continuous on $\mathbb{R}^3$ and vanishes outside a bounded region, show that, for any fixed $\mathbf{r}$,

$$\iiint_{\mathbb{R}^3} \frac{|f(\xi, \eta, \zeta)|}{|\mathbf{r} - \mathbf{s}|}\, d\xi\, d\eta\, d\zeta \le \text{constant}.$$

This shows that the potentials for the electric and magnetic fields corresponding to continuous charge and current densities that vanish outside bounded regions exist everywhere in $\mathbb{R}^3$. *Hint:* Without loss of generality you can assume $\mathbf{r} = \mathbf{0}$ and use spherical coordinates.

5. The electric charge density, ρ, in 3-space depends on time as well as position if charge is moving around. The motion is described by the current density, $\mathbf{J}$. Derive the **continuity equation**

$$\frac{\partial \rho}{\partial t} = -\mathbf{div}\, \mathbf{J}$$

from the fact that charge is conserved.

6. If $\mathbf{b}$ is a constant vector, show that

$$\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{b}|}\right) = -\frac{\mathbf{r} - \mathbf{b}}{|\mathbf{r} - \mathbf{b}|^3}.$$

7. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that for $\mathbf{r} \ne \mathbf{b}$,

$$\mathbf{div}\left(\mathbf{a} \times \frac{\mathbf{r} - \mathbf{b}}{|\mathbf{r} - \mathbf{b}|^3}\right) = 0.$$

Hint: Use identities (d) and (h) from Theorem 3 of Section 16.2.

8. Use the result of Exercise 7 to give an alternative proof that

$$\mathbf{div} \oint_{\mathcal{F}} \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3} = 0.$$

Note that **div** refers to the $\mathbf{r}$ variable.

9. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that for $\mathbf{r} \ne \mathbf{b}$,

$$\mathbf{curl}\left(\mathbf{a} \times \frac{\mathbf{r} - \mathbf{b}}{|\mathbf{r} - \mathbf{b}|^3}\right) = -(\mathbf{a} \bullet \nabla)\frac{\mathbf{r} - \mathbf{b}}{|\mathbf{r} - \mathbf{b}|^3}.$$

Hint: Use identity (e) from Theorem 3 of Section 16.2.

10. If $\mathbf{F}$ is any smooth vector field, show that

$$\oint_{\mathcal{F}} (d\mathbf{s} \bullet \nabla)\mathbf{F}(\mathbf{s}) = \mathbf{0}$$

around any closed loop $\mathcal{F}$. *Hint:* The gradients of the components of $\mathbf{F}$ are conservative.

11. Verify that if $\mathbf{r}$ does not lie on $\mathcal{F}$, then

$$\mathbf{curl} \oint_{\mathcal{F}} \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3} = \mathbf{0}.$$

Here, **curl** is taken with respect to the $\mathbf{r}$ variable.

12. Verify the formula $\mathbf{curl}\, \mathbf{A} = \mathbf{B}$, where $\mathbf{A}$ is the magnetic vector potential defined in terms of the steady-state current density $\mathbf{J}$.

13. If $\mathbf{A}$ is the vector potential for the magnetic field produced by a steady current in a closed-loop filament, show that $\mathbf{div}\, \mathbf{A} = 0$ off the filament.

14. If $\mathbf{A}$ is the vector potential for the magnetic field produced by a steady, continuous current density, show that $\mathbf{div}\, \mathbf{A} = 0$ everywhere. Hence, show that $\mathbf{A}$ satisfies the vector Poisson equation $\nabla^2\mathbf{A} = -\mathbf{J}$.

15. Show that in a region of space containing no charges ($\rho = 0$) and no currents ($\mathbf{J} = \mathbf{0}$), both $\mathbf{U} = \mathbf{E}$ and $\mathbf{U} = \mathbf{B}$ satisfy the wave equation

$$\frac{\partial^2 \mathbf{U}}{\partial t^2} = c^2 \nabla^2 \mathbf{U},$$

where $c = \sqrt{1/(\epsilon_0 \mu_0)} \approx 3 \times 10^8$ m/s.

16. As shown in this section, the static versions of Maxwell's equations needed revision when the fields **E** and **B** were allowed to depend on time. Show that the expression $\mathbf{E} = -\nabla\phi$ is no longer consistent with Maxwell's equations because the **E** field is no longer irrotational. Why does $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ continue to hold?

17. While the nonstatic Maxwell equations are not compatible with $\mathbf{E} = -\boldsymbol{\nabla}\phi$, show that they are compatible with the equation

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

18. **(Heat flow in 3-space)** The internal energy, E, of a volume element dV within a homogeneous solid is $\rho c T\, dV$, where ρ and c are constants (the density and specific heat of the solid material), and $T = T(x, y, z, t)$ is the temperature at time t at position (x, y, z) in the solid. Heat always flows in the direction of the negative temperature gradient and at a rate proportional to the size of that gradient. Thus, the rate of flow of heat energy across a surface element dS with normal $\hat{\mathbf{N}}$ is $-k\boldsymbol{\nabla}T \bullet \hat{\mathbf{N}}\, dS$, where k is also a constant depending on the material of the solid (the coefficient of thermal conductivity). Use "conservation of heat energy" to show that for any region R with surface $\mathcal{S}$ within the solid

$$\rho c \iiint_R \frac{\partial T}{\partial t}\, dV = k \oiint_{\mathcal{S}} \boldsymbol{\nabla}T \bullet \hat{\mathbf{N}}\, dS,$$

where $\hat{\mathbf{N}}$ is the unit outward normal on $\mathcal{S}$. Hence, show that heat flow within the solid is governed by the partial differential equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\boldsymbol{\nabla}^2 T = \frac{k}{\rho c}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$