THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM I: MAT300 Vector Analysis
DATE: 15. December 2016, 12:00 – 16:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII+
THE EXERCISE SHEET CONTAINS 3 EXERCISES ON 2 PAGES + 1 PAGE WITH FORMULAS

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1+\sin t)\,\mathbf{j} - \sqrt{2}\cos t\,\mathbf{k}\,,\quad 0 \le t \le 2\pi.$

a) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \pi$.

b) Compute the line integral

$$\int_{\mathscr{C}} y^2\,ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x,y,z) = (x^2 - z)\,\mathbf{i} - x^2\,\mathbf{j} + y\,\mathbf{k}\,.$$

c) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

d) Is the vector field $\mathbf{F}$ conservative? Give a reason to justify your answer.

EXERCISE 2

Let T be the solid region in the first octant that lies under the plane $2x + 2y + z = 2$.

a) Compute the triple integral

$$\iiint_T y\,dV\,.$$

Let $\mathscr{S}$ be the part of the plane $2x + 2y + z = 2$ that lies in the first octant.

b) Compute the surface integral

$$\iint_{\mathscr{S}} 2x\,dS\,.$$

EXERCISE 3

Consider the vector field $\mathbf{F}(x, y, z) = (x^3 - y^3)\,\mathbf{i} + (x^2 + x^3)\,\mathbf{j} + 3y^2 z\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{R}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane.
Let T be the solid region bounded by the surface $\mathscr{R}$ and the xy-plane.

b) Use the divergence theorem to compute the flux

$$\oiint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\,,$$

where $\mathscr{S}$ is the entire boundary surface of the region T, and $\hat{\mathbf{N}}$ is the unit normal vector field to $\mathscr{S}$, pointing outwards from T.

Let $\mathscr{C}$ be the circle in the xy-plane of radius 1, centred at the origin. The orientation on $\mathscr{C}$ is anticlockwise, when viewed from above.

c) Compute the line integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

Good luck!

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S}: z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}:\ G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}: z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}:\ G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla\bullet\mathbf{F}\ dV = \oiint\limits_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\ dS = \oint\limits_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

Alternate exam 15. December 2016, 12:00 – 16:00

Exercise 1 $\mathcal{C}: \underset{\sim}{r}(t) = \sin t\, \underset{\sim}{i} + (1+\sin t)\underset{\sim}{j} - \sqrt{2}\cos t\, \underset{\sim}{k},$
$0 \le t \le 2\pi.$

(a) $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \cos t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + \sqrt{2}\sin t\, \underset{\sim}{k}$

$\underset{\sim}{v}(\pi) = -\underset{\sim}{i} - \underset{\sim}{j}$ tangent vector to $\mathcal{C}$ at $\underset{\sim}{r}(\pi)$.

$|\underset{\sim}{v}(\pi)| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\Rightarrow \hat{\underset{\sim}{T}} = -\frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j}$ unit tangent vector to $\mathcal{C}$ at $\underset{\sim}{r}(\pi)$.

(b) $\int_{\mathcal{C}} y^2\, ds = \int_0^{2\pi} (1+\sin t)^2 \left|\frac{d\underset{\sim}{r}}{dt}\right| dt = \int_0^{2\pi} (1 + 2\sin t + \sin^2 t)\sqrt{2}\, dt$

$$\left[\ \left|\frac{d\underset{\sim}{r}}{dt}\right| = |\underset{\sim}{v}(t)| = \sqrt{\cos^2 t + \cos^2 t + (\sqrt{2}\sin t)^2} = \sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}\ \right]$$

use $\cos 2t = 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1}{2}(1 - \cos 2t)$

$$\int_{\mathcal{C}} y^2\, ds = \sqrt{2}\int_0^{2\pi} 1 + 2\sin t + \tfrac{1}{2} - \tfrac{1}{2}\cos 2t\ dt$$

$$= \sqrt{2}\left[\tfrac{3}{2}t - 2\cos t - \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= \sqrt{2}\cdot\tfrac{3}{2}\cdot 2\pi = 3\sqrt{2}\pi.$$

(c) $\underset{\sim}{F} = (x^2 - z)\underset{\sim}{i} - x^2 \underset{\sim}{j} + y \underset{\sim}{k}$

$$\underset{\sim}{F}(\underset{\sim}{r}(t)) = (\sin^2 t + \sqrt{2}\cos t)\underset{\sim}{i} - \sin^2 t \underset{\sim}{j} + (1 + \sin t)\underset{\sim}{k}$$

$$\underset{\sim}{F} \cdot \underset{\sim}{v} = \sin^2 t \cos t + \sqrt{2}\cos^2 t - \sin^2 t \cos t + \sqrt{2}\sin t + \sqrt{2}\sin^2 t$$

$$= \sqrt{2} + \sqrt{2}\sin t$$

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \underset{\sim}{F} \cdot \underset{\sim}{v}\, dt = \int_0^{2\pi} \sqrt{2} + \sqrt{2}\sin t \; dt$$

$$= \left(\sqrt{2}\,t - \sqrt{2}\cos t\right)\Big|_0^{2\pi} = 2\sqrt{2}\,\pi.$$

(d) Note that on $\mathcal{C}$, $\underset{\sim}{r}(0) = \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k} = (0, 1, -\sqrt{2})$
and $\underset{\sim}{r}(2\pi) = \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k} = (0, 1, -\sqrt{2})$.

Thus $\mathcal{C}$ is a closed curve; its two endpoints coincide.

If $\underline{F}$ were conservative, then we know that $\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r}$ would equal zero for every closed curve $\mathcal{C}$.

But since $\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} \neq 0$ for the given closed curve $\mathcal{C}$,

we know that $\underset{\sim}{F}$ is <u>not</u> <u>conservative</u>.

Alternative solution:

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - z & -x^2 & y \end{vmatrix} = \underline{i}(1) + \underline{j}(-1) + \underline{k}(-2x) \neq \underset{\sim}{0}$$

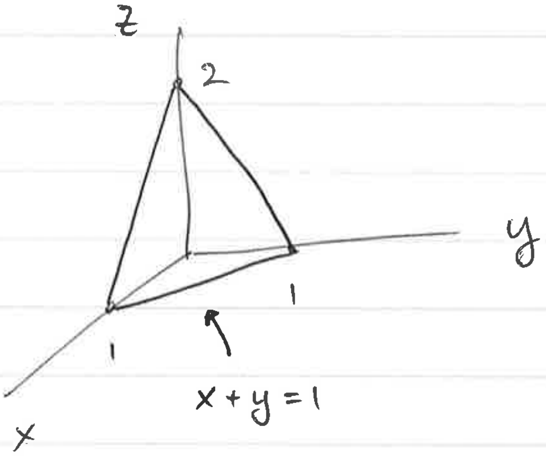
If $\underset{\sim}{F}$ were conservative, then it must be true that $\underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{0}$. Thus, since $\underset{\sim}{\nabla} \times \underset{\sim}{F} \neq \underset{\sim}{0}$, we see that $\underset{\sim}{F}$ is <u>not</u> <u>conservative</u>.

<u>Exercise 2</u> T: region under $2x+2y+z=2$, $x\geq 0, y\geq 0,$ $z\geq 0$.

(a) Let $x=0,\ y=0 \Rightarrow z=2$

$x=0,\ z=0 \Rightarrow y=1$

$y=0,\ z=0 \Rightarrow x=1$



T: $0 \leq y \leq 1$
$0 \leq x \leq 1-y$
$0 \leq z \leq 2-2x-2y$.

$$\iiint_T y\, dV = \int_0^1 dy \int_0^{1-y} dx \int_0^{2-2x-2y} y\, dz$$

$$= \int_0^1 dy \int_0^{1-y} dx\, [y(2-2x-2y)]$$

$$= \int_0^1 dy \int_0^{1-y} dx\, (2y - 2xy - 2y^2)$$

$$= \int_0^1 dy\, [2yx - x^2y - 2y^2x]_0^{1-y}$$

$$= \int_0^1 2y(1-y) - (1-y)^2y - 2y^2(1-y)\ dy$$

$$= \int_0^1 2y - 2y^2 - y + 2y^2 - y^3 - 2y^2 + 2y^3\ dy$$

$$= \int_0^1 y - 2y^2 + y^3\ dy = \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} = \frac{1}{12}.$$

(b) $S: 2x + 2y + z = 2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$

Let $G(x, y, z) = 2x + 2y + z$.

Then $S: G(x, y, z) = 2$, so S is a level surface of G, over the triangle R in the xy-plane,

$R: \quad 0 \leq y \leq 1, \quad 0 \leq x \leq 1 - y.$

$$\underset{\sim}{\nabla} G = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}, \qquad \frac{\partial G}{\partial z} = 1, \qquad \text{so}$$

$$dS = \frac{|\underset{\sim}{\nabla} G|}{\left|\frac{\partial G}{\partial z}\right|} dx\,dy = \frac{\sqrt{4+4+1}}{1} dx\,dy = 3\,dx\,dy.$$

Thus,
$$\iint_S 2x \, dS = \iint_R 2x \cdot 3 \, dx\,dy$$

$$= 3\int_0^1 dy \int_0^{1-y} 2x \, dx$$

$$= 3\int_0^1 dy \left[x^2 \right]_0^{1-y} = 3\int_0^1 (1-y)^2 dy$$

$$= 3\int_0^1 1 - 2y + y^2 \, dy = 3\left[y - y^2 + \tfrac{1}{3}y^3 \right]_0^1$$

$$= 3\left[1 - 1 + \tfrac{1}{3} \right] = 1.$$

Exercise 3 $\quad \underset{\sim}{F} = (x^3 - y^3)\underset{\sim}{i} + (x^2 + x^3)\underset{\sim}{j} + 3y^2 z \underset{\sim}{k}$

(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 3x^2 + 3y^2 = 3x^2 + 3y^2$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \frac{\partial}{\partial z} \\ x^3 - y^3 & x^2 + x^3 & 3y^2 z \end{vmatrix}$$

$$= \underset{\sim}{i}(6yz - 0) + \underset{\sim}{j}(0 - 0) + \underset{\sim}{k}(2x + 3x^2 + 3y^2)$$

(b) $R : z = 1 - x^2 - y^2, \quad z \geq 0.$

T : region bounded by R and $z = 0$.

$$I = \oiint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = \iiint_T (3x^2 + 3y^2)\, dV$$

dV — $r\, dr\, d\theta\, dz$

T: $0 \leq \theta \leq 2\pi$
$0 \leq r \leq 1$ $\qquad$ cylindrical coords
$0 \leq z \leq 1 - r^2$

$$I = \int_0^{2\pi} d\theta \int_0^1 dr \int_0^{1-r^2} 3r^2 \cdot r\, dz$$

$$= 2\pi \int_0^1 dr\; 3r^3(1 - r^2) = 6\pi \int_0^1 r^3 - r^5\, dr$$

$$= 6\pi \left[\frac{r^4}{4} - r^6/6 \right]_0^1 = 6\pi \left[1/4 - 1/6 \right] = 6\pi \cdot \frac{1}{12} = \frac{\pi}{2}.$$

(c) $\mathcal{C}$: $x^2+y^2=1$, $z=0$, anticlockwise (seen from above).

let $D=\{x^2+y^2\leq 1,\ z=0\}$, then if $\underset{\sim}{\hat{N}}=\underset{\sim}{k}$,

we have that the boundary of D is $\mathcal{C}$ with the induced orientation from D.

Stokes' theorem $\Rightarrow \oint_{\mathcal{C}} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \iint_D (\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot\underset{\sim}{\hat{N}}\,dS$

$$(\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot\underset{\sim}{\hat{N}} = (\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot\underset{\sim}{k} = 2x+3x^2+3y^2$$

$$\Rightarrow \oint_{\mathcal{C}} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \iint_D 2x+3x^2+3y^2\ dS \quad r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (2r\cos\theta + 3r^2)r\,dr$$

$$= \int_0^{2\pi} d\theta \left(\tfrac{2}{3}r^3\cos\theta + \tfrac{3}{4}r^4\right)\Big|_0^1$$

$$= \int_0^{2\pi} \tfrac{2}{3}\cos\theta + \tfrac{3}{4}\ d\theta$$

$$= \left[\tfrac{2}{3}\sin\theta + \tfrac{3}{4}\theta\right]_0^{2\pi} = \tfrac{3}{2}\pi.$$