THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
CONTINUATION EXAM: MAT300 Vector Analysis
DATE: 08.03.2017, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII+
THE EXERCISE SHEET CONTAINS 3 EXERCISES ON 2 PAGES + 1 PAGE WITH FORMULAS

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = (t-2)\,\mathbf{i} + (1+t)\,\mathbf{j} + (3-2t)\,\mathbf{k}\,, \quad 0 \le t \le 1$.

a) Find a unit tangent vector to $\mathscr{C}$.

b) Compute the line integral

$$\int_{\mathscr{C}} (x^2 + 2y + z)\, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x,y,z) = (2xyz + z)\,\mathbf{i} + x^2 z\,\mathbf{j} + (x^2 y + x - 2)\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.

d) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 2

Consider the transformation $x = u + v$, $y = u - 2v$, between the (x,y)-coordinates and the (u,v)-coordinates.
Let R be the bounded region in the xy-plane between the lines $y = x$, $y = x - 3$, $y = 3 - 2x$, and $y = -3 - 2x$.

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation.

b) Find the Jacobi determinants

$$\frac{\partial(x,y)}{\partial(u,v)} \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)}\,.$$

c) Use the change of coordinates given above to compute the double integral

$$\iint_R x^2 + x - y\, dA\,.$$

EXERCISE 3

Consider the vector field $\mathbf{F}(x, y, z) = (xz - xy^2)\,\mathbf{i} + 2x\,\mathbf{j} + (z - zx^2)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{R}$ be the part of the circular cylinder $x^2 + y^2 = 1$ that lies between the xy-plane and the plane $z = 3$.

Let T be the solid region bounded by $\mathscr{R}$, the xy-plane, and the plane $z = 3$.

b) Use the divergence theorem to compute the flux

$$\oiint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\,,$$

where $\mathscr{S}$ is the entire boundary surface of the region T, and $\hat{\mathbf{N}}$ is the unit normal vector field to $\mathscr{S}$, pointing outwards from T.

Let $\mathscr{C}$ be the circle in the xy-plane of radius 1, centred at the origin. The orientation on $\mathscr{C}$ is anticlockwise, when viewed from above.

c) Compute the line integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

Good luck!

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \,du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \,dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F} \bullet \hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_R \mathbf{F} \bullet \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_R \mathbf{F} \bullet \frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla \bullet \mathbf{F}\ dV = \oiint\limits_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla \times \mathbf{F}) \bullet \hat{\mathbf{N}}\ dS = \oint\limits_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla \bullet \mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

Second exam 2017

Exercise 1: $\mathcal{C}: \underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + (1+t)\underset{\sim}{j} + (3-2t)\underset{\sim}{k}, \quad 0 \le t \le 1.$

(a) $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, tangent vector to $\mathcal{C}$.

$$|\underset{\sim}{v}(t)| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\Rightarrow \hat{\underset{\sim}{T}} = \frac{\underset{\sim}{v}(t)}{|\underset{\sim}{v}(t)|} = \frac{1}{\sqrt{6}}\underset{\sim}{i} + \frac{1}{\sqrt{6}}\underset{\sim}{j} - \frac{2}{\sqrt{6}}\underset{\sim}{k}$$ unit tangent vector to $\mathcal{C}$.

(b) $$\int_{\mathcal{C}} (x^2 + 2y + z)\,ds = \int_0^1 \left[(t-2)^2 + 2(1+t) + (3-2t)\right] |\underset{\sim}{v}(t)|\,dt$$

$$= \sqrt{6}\int_0^1 t^2 - 4t + 4 + 2 + 2t + 3 - 2t \quad dt$$

$$= \sqrt{6}\int_0^1 t^2 - 4t + 9 \quad dt = \sqrt{6}\left(\tfrac{1}{3} - 2 + 9\right)$$

$$= \sqrt{6}\left(\frac{22}{3}\right) = \frac{22\sqrt{6}}{3}.$$

(c) $\underset{\sim}{F} = (2xyz + z)\underset{\sim}{i} + x^2z\underset{\sim}{j} + (x^2y + x - 2)\underset{\sim}{k}$

If $\underset{\sim}{F}$ is conservative then $\underset{\sim}{F} = \underset{\sim}{\nabla}\phi = \frac{\partial\phi}{\partial x}\underset{\sim}{i} + \frac{\partial\phi}{\partial y}\underset{\sim}{j} + \frac{\partial\phi}{\partial z}\underset{\sim}{k}$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xyz + z, \quad \frac{\partial\phi}{\partial y} = x^2 z, \quad \frac{\partial\phi}{\partial z} = x^2y + x - 2$$

$$① \qquad\qquad ② \qquad\qquad ③$$

$$① \Rightarrow \phi(x,y,z) = x^2yz + xz + f(y,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = x^2z + \frac{\partial f}{\partial y} \overset{②}{=} x^2z \Rightarrow f(y,z) = g(z)$$

$$\Rightarrow \phi = x^2yz + xz + g(z) \Rightarrow \frac{\partial\phi}{\partial z} = x^2y + x + \frac{\partial g}{\partial z} \overset{③}{=} x^2y + x - 2$$

$$\Rightarrow \frac{\partial g}{\partial z} = -2 \Rightarrow g(z) = -2z + c, \quad c \text{ constant.}$$

Thus $\phi(x,y,z) = x^2yz + xz - 2z + c$.

(d) Since $\underset{\sim}{F}$ is conservative with potential ϕ,

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}(1)) - \phi(\underset{\sim}{r}(0))$$

$$\phi(\underset{\sim}{r}(1)) = \phi(-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = \phi(-1,2,1) = 2 - 1 - 2 = -1$$
$$\phi(\underset{\sim}{r}(0)) = \phi(-2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}) = \phi(-2,1,3) = 12 - 6 - 6 = 0$$

$$\Rightarrow \int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = -1 - 0 = -1.$$

Exercise 2 $\quad x = u+v, \quad y = u-2v.$

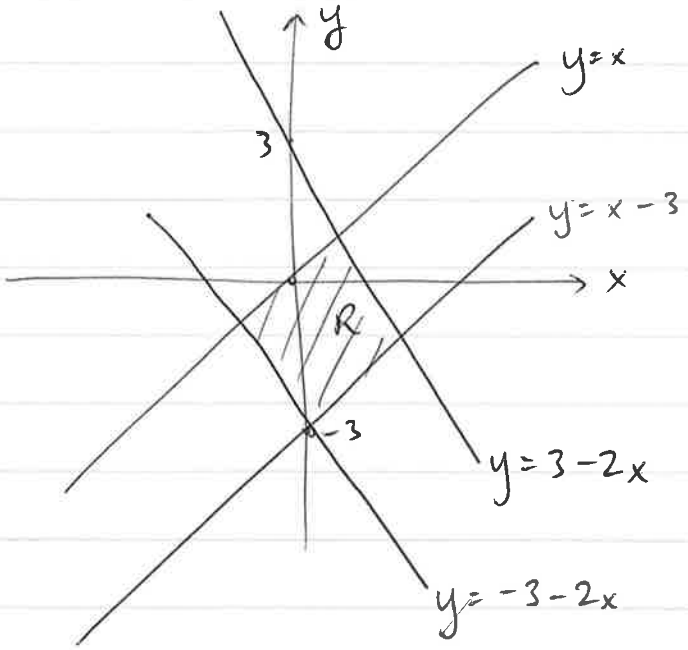
$R: \quad y = x, \quad y = x-3, \quad y = 3-2x, \quad y = -3-2x.$

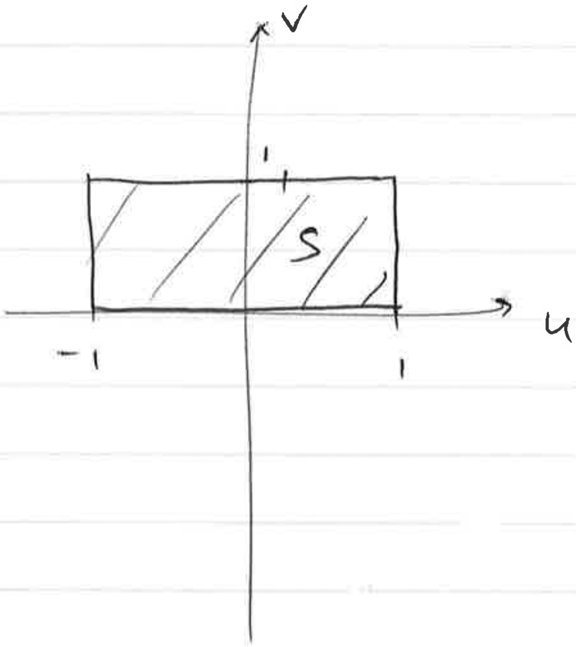
(a) $y = x$: $\quad u-2v = u+v \Rightarrow 3v = 0 \Rightarrow v = 0$

$y = x-3$: $\quad u-2v = u+v-3 \Rightarrow 3v = 3 \Rightarrow v = 1$

$y = 3-2x$: $\quad u-2v = 3-2u-2v \Rightarrow 3u = 3 \Rightarrow u = 1$

$y = -3-2x$: $\quad u-2v = -3-2u-2v \Rightarrow 3u = -3 \Rightarrow u = -1.$





(b) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -2-1 = -3$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = -\frac{1}{3}.$$

(c) $$\iint_R (x^2 + x - y)\,dx\,dy = \iint_S \left((u+v)^2 + u + v - u + 2v\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_S \left(u^2 + 2uv + v^2 + 3v\right) 3\,du\,dv$$

$$= 3\int_{-1}^{1} du \int_0^1 u^2 + 2uv + v^2 + 3v \quad dv$$

$$= 3\int_{-1}^{1} du \left(u^2 v + uv^2 + \tfrac{1}{3}v^3 + \tfrac{3}{2}v^2\right)\Big|_0^1$$

$$= 3\int_{-1}^{1} u^2 + u + \tfrac{1}{3} + \tfrac{3}{2} \quad du = 3\int_{-1}^{1} u^2 + u + \tfrac{11}{6} \quad du$$

$$= 3\left[\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + \tfrac{11}{6}u\right]_{-1}^{1}$$

$$= 3\left[\tfrac{1}{3} + \tfrac{1}{2} + \tfrac{11}{6} - \left(-\tfrac{1}{3} + \tfrac{1}{2} - \tfrac{11}{6}\right)\right]$$

$$= 3\left[\tfrac{2}{3} + \tfrac{11}{3}\right] = 13.$$

Exercise 3 $\quad \underset{\sim}{F} = (xz - xy^2)\underset{\sim}{i} + 2x\,\underset{\sim}{j} + (z - zx^2)\underset{\sim}{k}$

(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = z - y^2 + 0 + 1 - x^2 = 1 + z - x^2 - y^2$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz - xy^2 & 2x & z - zx^2 \end{vmatrix}$$

$$= \underset{\sim}{i}(0 - 0) + \underset{\sim}{j}(x + 2xz) + \underset{\sim}{k}(2 + 2xy)$$

(b) $R: \quad x^2 + y^2 = 1, \quad 0 \le z \le 3.$

T: region bounded by R, $z = 0$, $z = 3$.

In cylindrical coords, $T: \quad 0 \le \theta \le 2\pi$

$0 \le r \le 1$

$0 \le z \le 3$.

By div. theorem, $\displaystyle\oiint_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = \iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV$

$$= \iiint_T 1 + z - x^2 - y^2 \, dV \qquad r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr \int_0^3 (1 + z - r^2)\, r\, dz$$

$$= 2\pi \int_0^1 dr \left(zr + \tfrac{1}{2} z^2 r - zr^3 \right)\Big|_{z=0}^{3}$$

$$= 2\pi \int_0^1 \left(3r + \tfrac{9}{2} r - 3r^3 \right) dr$$

$$= 2\pi \left[\frac{15r^2}{4} - \tfrac{3}{4} r^4 \right]_0^1 = 2\pi \left[\frac{15}{4} - \tfrac{3}{4} \right] = 6\pi$$

(c) $\mathcal{C}$: $x^2 + y^2 = 1$, $z = 0$, anticlockwise (seen from above)

Let D: $x^2 + y^2 \leq 1$, $z = 0$. Then if $\hat{\underset{\sim}{N}} = \underset{\sim}{k}$, we have that $\mathcal{C}$ is the boundary of D with the induced orientation from D.

By Stokes' theorem, $\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} \, dS$.

$$(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} = (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{k} = 2 + 2xy$$

$$\Rightarrow \oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D 2 + 2xy \; dS \quad r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (2 + 2r^2 \cos\theta \sin\theta)\, r\, dr$$

$$= \int_0^{2\pi} d\theta \left[1 + \tfrac{1}{2} \cos\theta \sin\theta \right] \qquad \text{Use } \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \int_0^{2\pi} 1 + \tfrac{1}{4} \sin 2\theta \; d\theta = \left[\theta - \tfrac{1}{8} \cos 2\theta \right]_0^{2\pi}$$

$$= 2\pi.$$