THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM I: MAT300 Vector Analysis
DATE: 11. December 2017, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators permitted in accordance with TN faculty rules
THE EXERCISE SHEET CONTAINS 4 EXERCISES ON 2 PAGES + 1 PAGE WITH FORMULAS

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = (\sin t + t)\,\mathbf{i} - \sqrt{3}\sin t\,\mathbf{j} + (2\cos t + t)\,\mathbf{k}\,, \quad 0 \le t \le \frac{3\pi}{2}$.

a) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \pi$.

b) Compute the line integral

$$\int_{\mathscr{C}} z - \frac{2}{\sqrt{3}}y - x\,ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x,y,z) = (-z + y^2)\,\mathbf{i} + 2xy\,\mathbf{j} + (2z - x)\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.

d) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 2

Consider the transformation $u = x^2 + y$, $v = x - y$, between the (x,y)-coordinates and the (u,v)-coordinates.

Let R be the bounded region in the xy-plane between the lines $y = 1 - x^2$, $y = x$, and the y-axis.

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation.

b) Find the Jacobi determinants

$$\frac{\partial(x,y)}{\partial(u,v)} \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)}\,.$$

c) Use the change of coordinates given above to compute the double integral

$$\iint_R x(x+1)(2x+1)\,dA\,.$$

EXERCISE 3

Let $\mathscr{S}$ be the triangular surface with vertices at the points $(2, 0, 0), (0, 3, 0), (0, 0, 2)$. Let T be the solid region in the first octant that lies under the surface $\mathscr{S}$.

a) Give a brief reason to explain why

$$\iiint_T x^2 \, dV = \iiint_T z^2 \, dV \,,$$

and then compute the triple integral

$$\iiint_T x^2 + z^2 \, dV \,.$$

b) Compute the surface integral

$$\iint_{\mathscr{S}} 6 - 3z - 2y \, dS \,.$$

EXERCISE 4

Consider the vector field $\mathbf{F}(x, y, z) = (-2x^3 + y^3)\,\mathbf{i} + (3x^2y - y^3)\,\mathbf{j} + (z^3 + xy)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{S}$ be the piece of the surface of the cone defined by $z^2 = x^2 + y^2$, for $0 \leq z \leq 1$. Let $\hat{\mathbf{N}}$ be the downward pointing unit normal vector field to $\mathscr{S}$.

b) Use the divergence theorem to compute the flux

$$\iint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \, dS \,.$$

Let $\mathbf{G} = \mathbf{G}(x, y, z) = G_1(x, y, z)\,\mathbf{i} + G_2(x, y, z)\,\mathbf{j} + G_3(x, y, z)\,\mathbf{k}$ be a vector field, and let $z\mathbf{G}$ be the vector field given by multiplying each component of $\mathbf{G}$ by z, so that

$$z\mathbf{G}(x, y, z) = zG_1(x, y, z)\,\mathbf{i} + zG_2(x, y, z)\,\mathbf{j} + zG_3(x, y, z)\,\mathbf{k} \,.$$

c) Show that the curl of $z\mathbf{G}$ is given by the formula

$$\nabla \times (z\mathbf{G}) = z(\nabla \times \mathbf{G}) - G_2\,\mathbf{i} + G_1\,\mathbf{j} \,.$$

Hence, or otherwise, compute the flux integral

$$\iint_{\mathscr{S}} ((xz - 3x^2y + y^3)\,\mathbf{i} + (y^3 - 2x^3 - yz)\,\mathbf{j} + (6xyz - 3y^2z)\,\mathbf{k}) \bullet \hat{\mathbf{N}} \, dS \,.$$

Hint: First use the formula for $\nabla \times (z\mathbf{G})$ together your result from part (a) to compute $\nabla \times (z\mathbf{F})$, and then apply Stokes' theorem.

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| dt\,.$$

Line integral of a vector field $\mathbf{F}=F_1\,\mathbf{i}+F_2\,\mathbf{j}+F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}}ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1dx+F_2dy+F_3dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S}: z=g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}: G(x,y,z)=c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}: z=g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\pm(-\frac{\partial g}{\partial x}\,\mathbf{i}-\frac{\partial g}{\partial y}\,\mathbf{j}+\,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}: G(x,y,z)=c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla\bullet\mathbf{F}\ dV = \oint\!\!\!\!\!\iint\limits_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\ dS = \oint\limits_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i}+\frac{\partial}{\partial y}\,\mathbf{j}+\frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z)=(x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)=(x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = 2\cos^2\theta-1=1-2\sin^2\theta$.

1. $\mathcal{C}: \underset{\sim}{r}(t) = (\sin t + t)\underset{\sim}{i} - \sqrt{3}\sin t\, \underset{\sim}{j} + (2\cos t + t)\underset{\sim}{k}$

$0 \le t \le \frac{3\pi}{2}$.

(a) $\underset{\sim}{r}'(t) = (\cos t + 1)\underset{\sim}{i} - \sqrt{3}\cos t\, \underset{\sim}{j} + (-2\sin t + 1)\underset{\sim}{k}$

$\underset{\sim}{r}'(\pi) = (-1+1)\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + \underset{\sim}{k} = \sqrt{3}\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{r}'(\pi)| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$

$\Rightarrow \hat{\underset{\sim}{T}} = \dfrac{\underset{\sim}{r}'(\pi)}{|\underset{\sim}{r}'(\pi)|} = \underline{\underline{\frac{\sqrt{3}}{2}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}}}$

(b) $\displaystyle\int_{\mathcal{C}} z - \frac{2}{\sqrt{3}}y - x \, ds = \int_{\mathcal{C}} f(x,y,z)\,ds = \int_0^{\frac{3\pi}{2}} f(\underset{\sim}{r}(t))|r'(t)|\,dt$

$|\underset{\sim}{r}'(t)| = \sqrt{(\cos t + 1)^2 + (-\sqrt{3}\cos t)^2 + (-2\sin t + 1)^2}$

$= \sqrt{\cos^2 t + 2\cos t + 1 + 3\cos^2 t + 4\sin^2 t - 4\sin t + 1}$

$= \sqrt{6 + 2\cos t - 4\sin t}$

$f(\underset{\sim}{r}(t)) = 2\cos t + t - \frac{2}{\sqrt{3}}(-\sqrt{3}\sin t) - (\sin t + t)$

$= 2\cos t + \sin t$

So $\displaystyle\int_{\mathcal{C}} f\,ds = \int_0^{\frac{3\pi}{2}} (2\cos t + \sin t)(6 + 2\cos t - 4\sin t)^{\frac{1}{2}}\,dt.$

Let $u = 6 + 2\cos t - 4\sin t \Rightarrow du = (-2\sin t - 4\cos t)\,dt$

$= -2(\sin t + 2\cos t)\,dt$

$t = 0 \Rightarrow u = 8, \quad t = \frac{3\pi}{2} \Rightarrow u = 10.$

So $\int_{\mathcal{C}} f\, ds = \int_8^{10} -\frac{1}{2} u^{1/2}\, du = -\frac{1}{2}\cdot\frac{2}{3} u^{3/2}\Big|_8^{10} = \underline{-\frac{1}{3}\left[10^{3/2} - 8^{3/2}\right]}$

≈ -2.998

(c) $\underline{F}(x,y,z) = (-z+y^2)\underset{\sim}{i} + 2xy\underset{\sim}{j} + (2z-x)\underset{\sim}{k}$

$= \underset{\sim}{\nabla}\phi$

$\Rightarrow \frac{\partial\phi}{\partial x} \overset{①}{=} -z+y^2 \qquad \frac{\partial\phi}{\partial y} \overset{②}{=} 2xy \qquad \frac{\partial\phi}{\partial z} \overset{③}{=} 2z-x$

① $\Rightarrow \phi(x,y,z) = -zx + y^2x + c_1(y,z)$

$\Rightarrow \frac{\partial\phi}{\partial y} = 2yx + \frac{\partial c_1}{\partial y} \overset{②}{=} 2xy \Rightarrow c_1(y,z) = c_2(z).$

$\Rightarrow \phi(x,y,z) = -zx + y^2x + c_2(z)$

$\Rightarrow \frac{\partial\phi}{\partial z} = -x + \frac{\partial c_2}{\partial z} \overset{③}{=} 2z - x \Rightarrow c_2(z) = z^2 + c$

$\Rightarrow \phi(x,y,z) = \underline{\underline{-zx + y^2x + z^2 + c.}}$ (Can take $c=0$).

(d) $\int_{\mathcal{C}} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}(\frac{3\pi}{2})) - \phi(\underset{\sim}{r}(0))$ since $\underset{\sim}{F}$ is conservative with potential ϕ.

$\underline{r}(0) = (0,0,2)$, $\quad \underset{\sim}{r}(\frac{3\pi}{2}) = (\frac{3\pi}{2} - 1, \sqrt{3}, \frac{3\pi}{2})$.

$\phi(0,0,2) = 4$, $\quad \phi(\frac{3\pi}{2}-1, \sqrt{3}, \frac{3\pi}{2}) = -\frac{3\pi}{2}(\frac{3\pi}{2}-1) + 3(\frac{3\pi}{2}-1) + (\frac{3\pi}{2})^2$

$= \frac{3\pi}{2} + 3\cdot\frac{3\pi}{2} - 3 = 6\pi - 3$

$\Rightarrow \int_{\mathcal{C}} \underset{\sim}{F}\cdot d\underline{r} = 6\pi - 3 - 4 = \underline{\underline{6\pi - 7}}.$

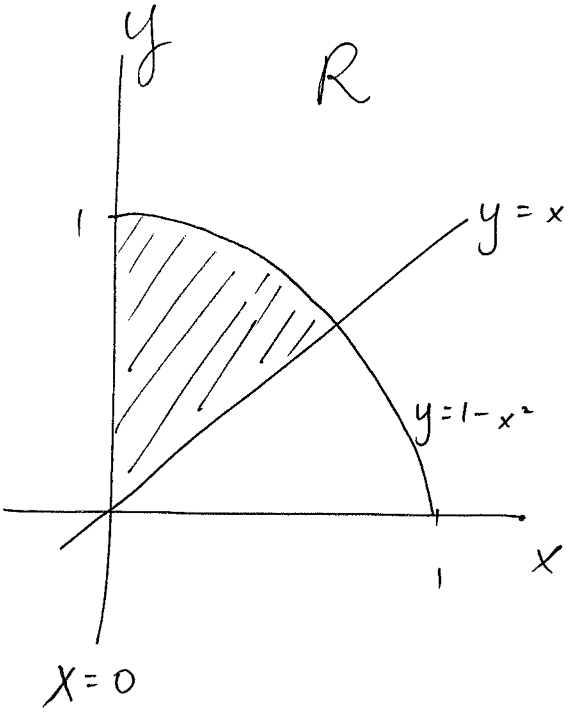
2. $u = x^2 + y$, $v = x - y$.

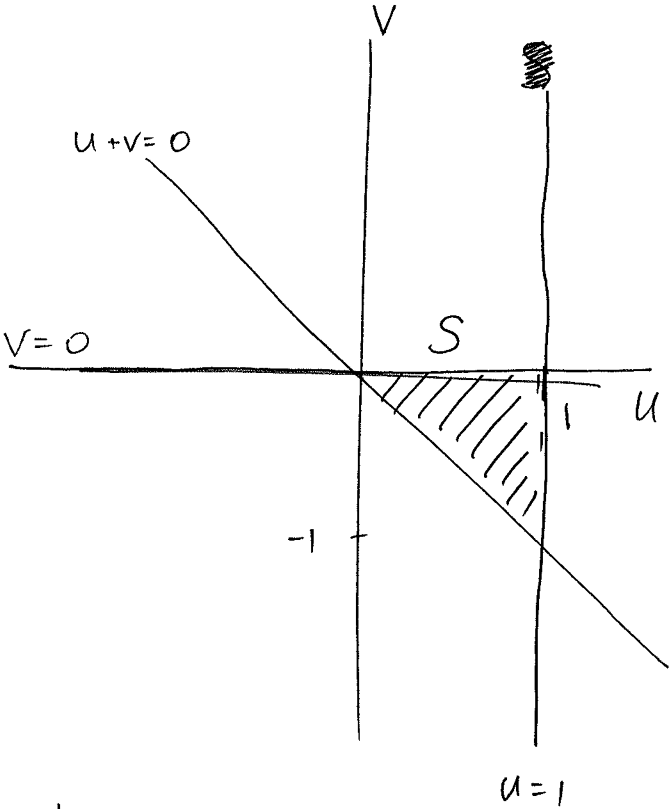
R: region between $y = 1 - x^2$, $y = x$, $x = 0$ (y-axis)

(a) $y = 1 - x^2 \Rightarrow x^2 + y = 1 \Rightarrow u = 1.$

$y = x \Rightarrow x - y = 0 \Rightarrow v = 0.$

$x = 0 \Rightarrow u + v = x^2 + y + x - y = x^2 + x = 0. \Rightarrow v = -u$





(b) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 1 \\ 1 & -1 \end{vmatrix} = -2x - 1 = -(2x+1).$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{-2x-1} = -\frac{1}{2x+1}.$$

(c)
$$I = \iint_R x(x+1)(2x+1)\,dA. \qquad dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{2x+1}\,du\,dv$$

(since $2x+1 \geq 0$ on R).

$$I = \iint_S \frac{(x^2+x)(2x+1)}{2x+1}\,du\,dv = \iint_S x^2+x \;du\,dv.$$

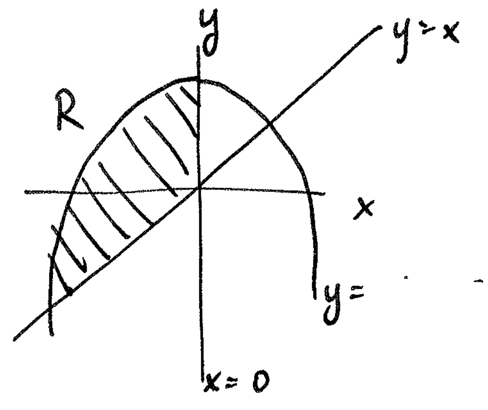
Now $u+v = x^2+x$, so $I = \iint_S u+v\;du\,dv.$

$S: 0 \leq u \leq 1, \quad -u \leq v \leq 0. \Rightarrow \quad I = \int_0^1 du \int_{-u}^0 u+v\;dv$

$$= \int_0^1 uv + \tfrac{1}{2}v^2 \Big|_{-u}^0 du = \int_0^1 -\left[-u^2 + \tfrac{1}{2}u^2\right] du = \int_0^1 \tfrac{1}{2}u^2\,du$$

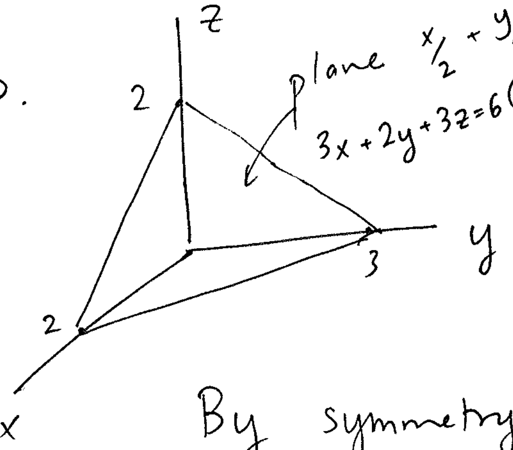
$$= \tfrac{1}{2}\cdot\tfrac{1}{3}u^3\Big|_0^1 = \underline{\underline{1/6}}$$

Note: The question should have stated: "Let R be the bounded region on the right half of the xy-plane ...". No marks are deducted if you sketched the region R on the left half plane bounded by the same three lines:



This does not affect the corresponding region S, nor the answers to part (b), (c).

3.



T : $0 \le x \le 2$

$0 \le z \le 2 - x$

$0 \le y \le 3 - \frac{3}{2}x - \frac{3}{2}z$

By symmetry of T under exchange of variables $x \longleftrightarrow z$, we have $\iiint_T x^2\, dV = \iiint_T z^2\, dV$.

We compute $\iiint_T x^2\, dV$ and multiply by 2 to get final answer.

$$\iiint_T x^2\, dV = \int_0^2 dx \int_0^{2-x} dz \int_0^{3-\frac{3}{2}x-\frac{3}{2}z} x^2\, dy$$

$$= \int_0^2 dx \int_0^{2-x} x^2\left(3 - \frac{3}{2}x - \frac{3}{2}z\right) dz = \int_0^2 3x^2z - \frac{3}{2}x^3z - \frac{3}{4}x^2z^2 \Big|_0^{2-x} dx$$

$$= \int_0^2 3x^2(2-x) - \frac{3}{2}x^3(2-x) - \frac{3}{4}x^2(4 - 4x + x^2)\, dx$$

$$= \int_0^2 6x^2 - 3x^3 - 3x^3 + \frac{3}{2}x^4 - 3x^2 + 3x^3 - \frac{3}{4}x^4\, dx$$

$$= \int_0^2 \frac{3}{4}x^4 - 3x^3 + 3x^2\, dx = \frac{3}{20}\cdot 2^5 - \frac{3}{4}\cdot 2^4 + 2^3$$

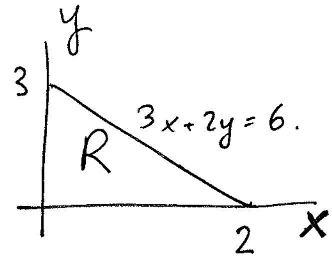
$$= \frac{24}{5} - 12 + 8 = \frac{4}{5}.$$

So $\iiint_T x^2 + z^2\, dV = 2\iiint_T x^2\, dV = \underline{8/5}$.

(b) $\iint_S 6 - 3z - 2y \, dS$.

$S: \; 3x + 2y + 3z = 6, \quad x \geq 0, \; y \geq 0, \; z \geq 0$

Projection onto xy-plane ~~is~~ is R:



Let $G(x,y,z) = 3x + 2y + 3z$, then $S: G(x,y,z) = 6$.

$$\Rightarrow dS = \left| \frac{\nabla G}{\frac{\partial G}{\partial z}} \right| dx\,dy = \frac{|3\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}|}{3} dx\,dy = \frac{\sqrt{9+4+9}}{3} dx\,dy$$

$$= \frac{\sqrt{22}}{3} dx\,dy.$$

On S, $6 - 3z - 2y = 6 - (6 - 2y - 3x) - 2y$

$$= 3x$$

So $\iint_S 6 - 3z - 2y \, dS = \iint_R 3x \frac{\sqrt{22}}{3} dx\,dy = \sqrt{22} \int_0^2 \int_0^{3 - \frac{3}{2}x} x \, dy\,dx$

$$= \sqrt{22} \int_0^2 x\left(3 - \frac{3}{2}x\right) dx = \sqrt{22} \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_0^2$$

$$= \sqrt{22}[6 - 4] = \underline{\underline{2\sqrt{22}}}.$$

4. $\underline{F}(x,y,z) = (-2x^3+y^3)\underset{\sim}{i} + (3x^2y - y^3)\underset{\sim}{j} + (z^3+xy)\underset{\sim}{k}$

(a) $\underset{\sim}{\nabla}\cdot\underset{\sim}{F} = -6x^2 + 3x^2 - 3y^2 + 3z^2 = \underline{3(z^2-y^2-x^2)}$

$$\underset{\sim}{\nabla}\times\underset{\sim}{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2x^3+y^3 & 3x^2y-y^3 & z^3+xy \end{vmatrix} = \underset{\sim}{i}(x) + \underline{j}(-y) + \underline{k}(6xy-3y^2)$$

$$= \underline{x\underline{i} - y\underline{j} + (6xy-3y^2)\underset{\sim}{k}}.$$

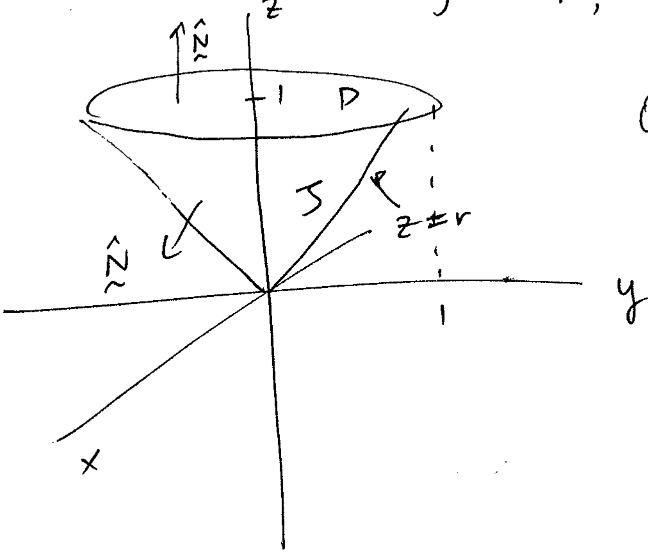
(b) $S: z^2 = x^2+y^2, \quad 0 \le z \le 1, \quad \underset{\sim}{\hat{N}}$ downwards.

$$\iint_S \underset{\sim}{F}\cdot\underset{\sim}{\hat{N}}\,dS + \iint_D \underset{\sim}{F}\cdot\underset{\sim}{\hat{N}}\,dS = \iiint_T \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\,dV,$$ by the

divergence theorem, where T is solid cone

$T: x^2+y^2 \le z^2, \quad 0 \le z \le 1$, and D is disc

$D: x^2+y^2 \le 1, \quad z = 1$, with $\underset{\sim}{\hat{N}} = \underset{\sim}{k}$.



On D, $\underset{\sim}{F}\cdot\underset{\sim}{\hat{N}} = \underset{\sim}{F}\cdot\underset{\sim}{k} = z^3 + xy = xy + 1$ (since D is in $z=1$ plane).

$$\iint_D \underset{\sim}{F}\cdot\underset{\sim}{\hat{N}}\,dS = \iint_D xy+1\,dx\,dy$$

$$= \iint_D 1\,dx\,dy$$ due to symmetry of D and xy being odd in x, say.

$= \text{area}(D) = \pi$.

Now $T: \ 0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad r \le z \le 1$ in cyl. coords.

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 3(z^2 - y^2 - x^2) = 3(z^2 - r^2) \text{ in cyl. coords.}$$

$$\iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \, dV = 3 \int_0^{2\pi} d\theta \int_0^1 dr \int_r^1 (z^2 - r^2)\, r\, dz$$

$$= 3 \cdot 2\pi \cdot \int_0^1 \tfrac{1}{3} z^3 r - r^3 z \Big|_r^1 \, dr$$

$$= 6\pi \int_0^1 \tfrac{1}{3} r - r^3 - \tfrac{1}{3} r^4 + r^4 \, dr$$

$$= 6\pi \int_0^1 \tfrac{2}{3} r^4 - r^3 + \tfrac{1}{3} r \, dr$$

$$= 6\pi \left[\frac{2}{15} - \frac{1}{4} + \frac{1}{6} \right] = 6\pi \left[\frac{8 - 15 + 10}{60} \right] = \frac{3\pi}{10}.$$

So $\displaystyle \iint_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS = \frac{3\pi}{10} - \pi = \underline{\underline{-\frac{7\pi}{10}}}.$

(c) $\underset{\sim}{\nabla} \times (z \underset{\sim}{G}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z G_1 & z G_2 & z G_3 \end{vmatrix}$

$$= \underline{i}\left(\frac{\partial}{\partial y}(z G_3) - \frac{\partial}{\partial z}(z G_2)\right) + \underline{j}\left(\frac{\partial}{\partial z}(z G_1) - \frac{\partial}{\partial x}(z G_3)\right)$$
$$+ \underline{k}\left(\frac{\partial}{\partial x}(z G_2) - \frac{\partial}{\partial y}(z G_1)\right)$$

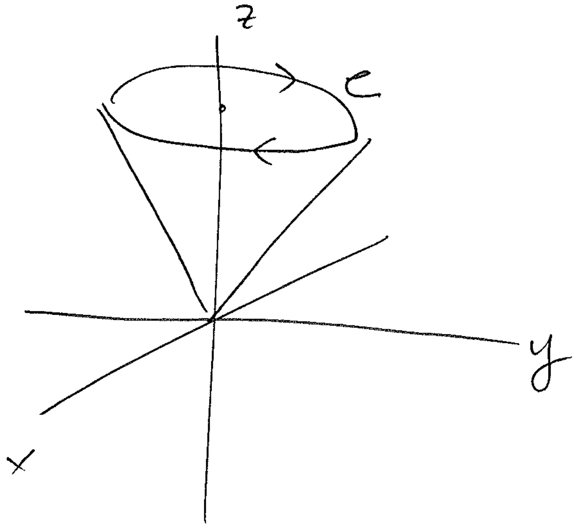
$$= \underset{\sim}{i}\left(z\frac{\partial G_3}{\partial y} - G_2 - z\frac{\partial G_2}{\partial z}\right) + \underline{j}\left(G_1 + z\frac{\partial G_1}{\partial z} - z\frac{\partial G_3}{\partial x}\right)$$
$$+ \underset{\sim}{k}\left(z\frac{\partial G_2}{\partial x} - z\frac{\partial G_1}{\partial y}\right) \text{ by product rule}$$

$$= z(\underset{\sim}{\nabla} \times \underset{\sim}{G}) - G_2 \underset{\sim}{i} + G_1 \underline{j} \quad \text{as required.}$$

$$\underset{\sim}{\nabla} \times (z \underset{\sim}{F}) = z(\underline{\nabla} \times \underline{F}) - F_2 \underline{i} + F_1 \underline{j}$$
$$= zx\underline{i} - zy\underline{j} + (6xyz - 3y^2 z)\underline{k} - (3x^2 y - y^3)\underset{\sim}{i}$$
$$+ (-2x^3 + y^3)\underline{j}$$
$$= \underset{\sim}{i}(zx - 3x^2 y + y^3) + \underline{j}(-2x^3 + y^3 - zy) + \underline{k}(6xyz - 3y^2 z).$$

So we need to compute $I = \iint_S \underset{\sim}{\nabla} \times (z\underline{F}) \cdot \underset{\sim}{\hat{N}}\, dS$.

By Stokes' theorem $= \oint_C z\underset{\sim}{F} \cdot d\underset{\sim}{r}$, where C has clockwise orientation when viewed from above.



But on C, $z = 1$ everywhere, so get

$I = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$. Apply Stokes' theorem again to disc D with $\hat{\underset{\sim}{N}} = -\underset{\sim}{k}$, whose boundary is also C;

$$\Rightarrow \quad I = \iint_D \underset{\sim}{\nabla} \times (z \underset{\sim}{F}) \cdot (-\underset{\sim}{k})\, dx\, dy.$$

$$= -\iint_D 6xyz - 3y^2 z \, dx\,dy = \iint_D 3y^2 - 6xy \, dx\,dy$$

$(z = 1$ on $D)$

$$= 3 \iint_D y^2 \, dx\,dy$$

since D is symmetric, under reflection $x \to -x$ and xy is odd

$$= 3 \int_0^{2\pi} \int_0^1 r^2 \sin^2\theta \, r \, dr\, d\theta = \frac{3}{4} \int_0^{2\pi} \underbrace{\sin^2\theta}_{\frac{1}{2}(1-\cos 2\theta)} \, d\theta$$

$$= \frac{3}{4} \left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \underline{\underline{\frac{3\pi}{4}}}.$$

Alternative solution: from $I = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

Compute directly: $x = \cos\theta, \quad y = -\sin\theta, \quad \theta \in [0, 2\pi]$

$dx = -\sin\theta\, d\theta, \quad dy = -\cos\theta\, d\theta, \quad dz = 0$

$$\underset{\sim}{F}(\cos\theta, -\sin\theta, 1) = (-2\cos^3\theta - \sin^3\theta)\underset{\sim}{i} + (-3\cos^2\theta\sin\theta + \sin^3\theta)\underset{\sim}{j} + (1 - \cos\theta\sin\theta)\underset{\sim}{k}$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = [\underset{①}{2\cos^3\theta\sin\theta} + \underset{②}{\sin^4\theta} + \underset{③}{3\cos^3\theta\sin\theta} - \underset{④}{\sin^3\theta\cos\theta}]\, d\theta$$

Terms ①, ②, ④ vanish when integrated from $\theta = 0$ to 2π,

for example, $\int_0^{2\pi} \cos^3\theta \sin\theta\, d\theta = -\frac{1}{4}\cos^4\theta \Big|_0^{2\pi} = 0.$

$$\sin^4\theta = (\sin^2\theta)^2 = \left(\frac{1}{2}(1 - \cos 2\theta)\right)^2 = \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$$

$\cos 2\theta$ will integrate to 0. $\quad \cos^2 2\theta = \frac{1}{2}(\cos 4\theta + 1)$

will integrate to 0

$$\Rightarrow \int_0^{2\pi} \sin^4\theta\, d\theta = \frac{1}{4} \cdot \int_0^{2\pi} 1 + \frac{1}{2}\, d\theta$$

$$= \frac{3}{8} \cdot 2\pi = \underline{\frac{3}{4}\pi}.$$