THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
SAMPLE EXAM QUESTIONS: MAT300 Vector Analysis
DATE: December 2017, exam length is 4 hours
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators: Permitted calculators only
THE EXERCISE SHEET CONTAINS 8 EXERCISES (the actual exam will be shorter) + 1 PAGE WITH FORMULAS

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = 3t\,\mathbf{i} + (2+4t)\,\mathbf{j} - 5t\,\mathbf{k}\,, \quad 0 \le t \le 1.$

a) Find a unit tangent vector to $\mathscr{C}$.
b) Compute the line integral

$$\int_{\mathscr{C}} (x^2 + 2y + z)\, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x,y,z) = (z^2 - y)\,\mathbf{i} + (2y - x)\,\mathbf{j} + 2xz\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.
d) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 2

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = e^t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 2\sin t\,\mathbf{k}\,, \quad 0 \le t \le \pi.$

a) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \pi/2$.
b) Compute the line integral

$$\int_{\mathscr{C}} (x^2 + y^2 + z^2 - 4)\, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x,y,z) = (yz^2 + 2)\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.
d) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 3

Consider the curve $\mathscr{C}_1$: $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + (t/\pi)\,\mathbf{k}\,, \quad 0 \le t \le \pi.$

a) Compute the line integral

$$\int_{\mathscr{C}_1} x^2 + xy + z^2 \, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} - \pi z\,\mathbf{j} + (x - \pi xy)\,\mathbf{k}\,.$$

b) Compute the line integral

$$\int_{\mathscr{C}_1} \mathbf{F} \bullet d\mathbf{r}\,.$$

Now consider the curve $\mathscr{C}_2 : \mathbf{r}(t) = (1 - 2t)\,\mathbf{j} + t\,\mathbf{k}\,, \quad 0 \le t \le 1.$

c) Compute the line integral

$$\int_{\mathscr{C}_2} \mathbf{F} \bullet d\mathbf{r}\,.$$

d) Is the vector field $\mathbf{F}$ conservative? Give a reason to justify your answer.

EXERCISE 4

Consider the transformation $x = u - v$, $y = 2u + v$, between the (x, y)-coordinates and the (u, v)-coordinates.

Let R be the bounded region in the xy-plane between the lines $y = -x$, $y = 6 - x$, $y = 2x$, and $y = 3 + 2x$.

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation.

b) Find the Jacobi determinants

$$\frac{\partial(x, y)}{\partial(u, v)} \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)}\,.$$

c) Use the change of coordinates given above to compute the double integral

$$\iint_R (x + y)(y - 2x + 1)\, dA\,.$$

EXERCISE 5

Let $\mathscr{S}$ be the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.
Let T be the solid region between the surface $\mathscr{S}$ and the xy-plane.

a) Compute the triple integral

$$\iiint_T z + x^2 + y^2 \, dV \, .$$

b) Compute the surface integral

$$\iint_{\mathscr{S}} 4z - 17 \, dS \, .$$

EXERCISE 6

Let T be the solid region in the first octant that lies under the plane $3x + y + 3z = 6$.

a) Compute the triple integral

$$\iiint_T x \, dV \, .$$

Let $\mathscr{S}$ be the part of the plane $3x + y + 3z = 6$ that lies in the first octant.

b) Compute the surface integral

$$\iint_{\mathscr{S}} x(3z + y) \, dS \, .$$

EXERCISE 7

Consider the vector field $\mathbf{F}(x, y, z) = (z - x)\,\mathbf{i} + (3y + z^2)\,\mathbf{j} + x^2 y\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let T be the solid region in the first octant bounded by the plane $2y + z = 4$ and the plane $x = 4$.

b) Use the divergence theorem to compute the flux

$$\oiint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}} \, dS \, ,$$

where $\mathscr{S}$ is the entire boundary surface of the region T, and $\hat{\mathbf{N}}$ is the unit normal vector field to $\mathscr{S}$, pointing outwards from T.

Let $\mathscr{C}$ be the rectangular path with vertices at $(0, 0, 4)$, $(4, 0, 4)$, $(4, 2, 0)$ and $(0, 2, 0)$. The orientation on $\mathscr{C}$ is anticlockwise, when viewed from above.

c) Compute the line integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r} \, .$$

EXERCISE 8

Consider the vector field $\mathbf{F}(x,y,z) = (x^2+y)\,\mathbf{i} + (yz-y-x^2)\,\mathbf{j} + z\sqrt{x^2+y^2}\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{R}$ be the part of the circular cylinder $x^2+y^2=4$ that lies between the xy-plane and the plane $z=2$.

Let T be the solid region bounded by $\mathscr{R}$, the xy-plane, and the plane $z=2$.

b) Use the divergence theorem to compute the flux

$$\oiint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\,,$$

where $\mathscr{S}$ is the entire boundary surface of the region T, and $\hat{\mathbf{N}}$ is the unit normal vector field to $\mathscr{S}$, pointing outwards from T.

Let $\mathscr{C}$ be the circle in the plane $z=2$ of radius 2, centred at the point $(0,0,2)$. The orientation on $\mathscr{C}$ is anticlockwise, when viewed from above.

c) Compute the line integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

Good luck!

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \,du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \,dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F} \bullet \hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}) dt\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\; dS = \iint_R f \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\; dS = \iint_R f \frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_R \mathbf{F} \bullet \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_R \mathbf{F} \bullet \frac{\pm \nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla \bullet \mathbf{F}\; dV = \oint\!\!\!\oint\limits_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\; dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla \times \mathbf{F}) \bullet \hat{\mathbf{N}}\; dS = \oint\limits_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla \bullet \mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla \times \mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.