# Sample Exam Questions

**Exercise 1** $\quad \mathcal{C}: \ \underset{\sim}{r}(t) = 3t\,\underset{\sim}{i} + (2+4t)\,\underset{\sim}{j} - 5t\,\underset{\sim}{k}, \quad 0 \le t \le 1.$

(a) $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt}(t) = 3\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$ is tangent to $\mathcal{C}$, but not unit length.

$$|\underset{\sim}{v}(t)| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$$

$\Rightarrow$ unit tangent $\hat{\underset{\sim}{T}} = \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = \dfrac{1}{5\sqrt{2}}(3\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}).$

(b) $$\int_{\mathcal{C}} (x^2 + 2y + z)\, ds = \int_0^1 \left[(3t)^2 + 2(2+4t) - 5t\right] \cdot \underbrace{5\sqrt{2}}_{|\underset{\sim}{v}(t)|}\, dt$$

$$= 5\sqrt{2}\int_0^1 9t^2 + 4 + 8t - 5t \, dt = 5\sqrt{2}\int_0^1 4 + 3t + 9t^2 \, dt$$

$$= 5\sqrt{2}\left[4 + \tfrac{3}{2} + 3\right] = \frac{85\sqrt{2}}{2} = \frac{85}{\sqrt{2}}.$$

(c) $\underset{\sim}{F} = (z^2 - y)\,\underset{\sim}{i} + (2y - x)\,\underset{\sim}{j} + 2xz\,\underset{\sim}{k}$

$\underset{\sim}{F}$ conservative $\Leftrightarrow$ $\underset{\sim}{F} = \nabla\phi = \partial\phi/\partial x\,\underset{\sim}{i} + \partial\phi/\partial y\,\underset{\sim}{j} + \dfrac{\partial\phi}{\partial z}\,\underset{\sim}{k}.$

$$\Rightarrow \frac{\partial\phi}{\partial x} = z^2 - y, \ (1) \qquad \frac{\partial\phi}{\partial y} = 2y - x, \ (2) \qquad \frac{\partial\phi}{\partial z} = 2xz. \ (3)$$

By (1), $\phi(x,y,z) = z^2x - yx + f(y,z).$

Then $\frac{\partial \phi}{\partial y} = -x + \frac{\partial f}{\partial y} \overset{(2)}{=} 2y - x \Rightarrow \frac{\partial f}{\partial y} = 2y \Rightarrow f(y,z) = y^2 + g(z)$

So $\phi = z^2 x - yx + y^2 + g(z)$

$\Rightarrow \frac{\partial \phi}{\partial z} = 2zx + \frac{\partial g}{\partial z} \overset{(3)}{=} 2xz \Rightarrow g(z) = \text{constant} = c.$

Thus $\phi(x,y,z) = xz^2 - xy + y^2 + c.$

[Check: $\frac{\partial \phi}{\partial x} = z^2 - y$ ✓ $\frac{\partial \phi}{\partial y} = -x + 2y$ ✓ $\frac{\partial \phi}{\partial z} = 2xz$ ✓ ]

(d) Since $\underset{\sim}{F}$ is conservative with scaler potential $\phi$, we know that

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}(1)) - \phi(\underset{\sim}{r}(0))$$

$= \phi(3\underset{\sim}{i} + 6\underset{\sim}{j} - 5\underset{\sim}{k}) - \phi(2\underset{\sim}{j})$

$= \phi(3, 6, -5) - \phi(0, 2, 0) = 75 - 18 + 36 - 4 = 89$

Exercise 2 $\quad \mathcal{C}: \underset{\sim}{r}(t) = e^t \underset{\sim}{i} + 2\cos t \underset{\sim}{j} + 2\sin t \underset{\sim}{k}, \quad 0 \le t \le \pi.$

(a) $\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt}(t) = e^t \underset{\sim}{i} - 2\sin t \underset{\sim}{j} + 2\cos t \underset{\sim}{k}.$

At $t = \pi/2$, $\quad \underset{\sim}{v}(\pi/2) = e^{\pi/2}\underset{\sim}{i} - 2\underset{\sim}{j}$, tangent to $\mathcal{C}$ at $\underset{\sim}{r}(\pi/2)$.

$$|\underset{\sim}{v}(\pi/2)| = \sqrt{(e^{\pi/2})^2 + (-2)^2} = \sqrt{e^{\pi} + 4}$$

$\Rightarrow$ unit tangent to $\mathcal{C}$ at $\underset{\sim}{r}(\pi/2)$ is

$$\frac{\underset{\sim}{v}(\pi/2)}{|\underset{\sim}{v}(\pi/2)|} = \frac{1}{\sqrt{e^{\pi}+4}}\left(e^{\pi/2}\underset{\sim}{i} - 2\underset{\sim}{j}\right).$$

(b) Need $\quad \left|\frac{d\underset{\sim}{r}(t)}{dt}\right| = |\underset{\sim}{v}(t)| = \sqrt{(e^t)^2 + (-2\sin t)^2 + (2\cos t)^2}$

$$= \sqrt{e^{2t} + 4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{e^{2t} + 4}$$

Then $\displaystyle\int_{\mathcal{C}} x^2 + y^2 + z^2 - 4 \, ds = \int_0^{\pi} (e^{2t} + 4\cos^2 t + 4\sin^2 t - 4)\sqrt{e^{2t}+4}\, dt$

$$= \int_0^{\pi} e^{2t}\sqrt{e^{2t}+4}\, dt = \frac{2}{3}\cdot\frac{1}{2}\left(e^{2t}+4\right)^{3/2}\Big|_0^{\pi}$$

$$= \frac{1}{3}\left[\left(e^{2\pi}+4\right)^{3/2} - (1+4)^{3/2}\right] = \frac{1}{3}\left[\left(e^{2\pi}+4\right)^{3/2} - 5^{3/2}\right].$$

(c) $\underset{\sim}{F} = (yz^2+2)\underset{\sim}{i} + xz^2\underset{\sim}{j} + 2xyz\,\underset{\sim}{k}$

$\underset{\sim}{F}$ conservative $\iff$ $\underset{\sim}{F} = \underset{\sim}{\nabla}\phi$ for some $\phi$

$$\iff \frac{\partial\phi}{\partial x} = yz^2+2, \quad \frac{\partial\phi}{\partial y} = xz^2, \quad \frac{\partial\phi}{\partial z} = 2xyz.$$
$$\quad\quad ① \quad\quad\quad\quad ② \quad\quad\quad\quad ③$$

$$① \Rightarrow \phi = xyz^2 + 2x + f(y,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = xz^2 + \frac{\partial f}{\partial y} \overset{②}{=} xz^2 \Rightarrow f(y,z) = g(z)$$

$$\Rightarrow \phi = xyz^2 + 2x + g(z) \Rightarrow \frac{\partial\phi}{\partial z} = 2xyz + \frac{\partial g}{\partial z} \overset{③}{=} 2xyz$$

$\Rightarrow g(z) = c$ constant.

Thus $\phi(x,y,z) = xyz^2 + 2x + c$.

(d) Since $\underset{\sim}{F}$ is conservative with potential $\phi$,

$$\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}(\pi)) - \phi(\underset{\sim}{r}(0)) = \phi(e^{\pi}, -2, 0) - \phi(1,2,0)$$
$$= 2e^{\pi} - 2$$

Exercise 3 $\quad \mathcal{C}_1: \ \underset{\sim}{r}_1(t) = \sin t \ \underset{\sim}{i} + \cos t \ \underset{\sim}{j} + (t/\pi) \ \underset{\sim}{k}, \quad 0 \le t \le \pi.$

(a) Need $\underset{\sim}{v}(t) = \cos t \ \underset{\sim}{i} - \sin t \ \underset{\sim}{j} + 1/\pi \ \underset{\sim}{k}$

$$\Rightarrow \quad |\underset{\sim}{v}(t)| = \sqrt{\cos^2 t + (-\sin t)^2 + (1/\pi)^2}$$

$$= \sqrt{1 + (1/\pi)^2}$$

$$= \frac{\sqrt{1+\pi^2}}{\pi}$$

$|\underset{\sim}{v}(t)|$

$$\int_{\mathcal{C}_1} (x^2 + xy + z^2)\, ds = \int_0^\pi \left(\sin^2 t + \sin t \cos t + \frac{t^2}{\pi^2}\right) \cdot \frac{\sqrt{1+\pi^2}}{\pi}\, dt$$

Use $\quad \sin^2 t = \frac{1}{2}(1 - \cos 2t), \quad \sin t \cos t = \frac{1}{2} \sin 2t$

$$= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t + \frac{t^2}{\pi^2}\right) \frac{\sqrt{1+\pi^2}}{\pi}\, dt$$

$$= \frac{\sqrt{1+\pi^2}}{\pi}\left[\frac{t}{2} - \frac{1}{4}\sin 2t - \frac{1}{4}\cos 2t + \frac{t^3}{3\pi^2}\right]_0^\pi$$

$$= \frac{\sqrt{1+\pi^2}}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{3}\right] = \frac{5\sqrt{1+\pi^2}}{6}$$

$$\underset{\sim}{F} = x\underset{\sim}{i} - \pi z\underset{\sim}{j} + (x - \pi xy)\underset{\sim}{k}$$

(b) $\underset{\sim}{F}(\underset{\sim}{r}(t)) = \sin t\,\underset{\sim}{i} - t\,\underset{\sim}{j} + (\sin t - \pi \sin t \cos t)\underset{\sim}{k}$

$$\underset{\sim}{F} \cdot \underset{\sim}{v} = \sin t \cos t + t \sin t + \tfrac{1}{\pi} \sin t - \sin t \cos t$$

$$= t \sin t + \tfrac{1}{\pi} \sin t$$

$$\int_{C_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{\pi} \left(t \sin t + \tfrac{1}{\pi} \sin t\right) dt$$

Now let $u = t \Rightarrow du = dt$, $v = -\cos t \Rightarrow dv = \sin t\, dt$

Then $\int t \sin t\, dt = \int u\, dv = uv - \int v\, du = -t\cos t + \int \cos t\, dt$

$$= -t\cos t + \sin t$$

So $\displaystyle\int_0^{\pi} t \sin t + \tfrac{1}{\pi} \sin t\, dt = -t\cos t + \sin t - \tfrac{1}{\pi}\cos t \Big|_0^{\pi}$

$$= -\pi(-1) + 0 - \frac{1}{\pi}(-1) - \left[0 + 0 - \frac{1}{\pi}\right]$$

$$= \pi + \frac{1}{\pi} + \frac{1}{\pi} = \pi + \frac{2}{\pi}.$$

(c) $C_2$: $\underset{\sim}{r}(t) = (1 - 2t)\underset{\sim}{j} + t\underset{\sim}{k}$, $0 \le t \le 1$.

$$\underset{\sim}{v}(t) = -2\underset{\sim}{j} + \underset{\sim}{k},$$

$\underset{\sim}{F}(\underset{\sim}{r}(t)) = -\pi t\,\underset{\sim}{j}$, $\quad \underset{\sim}{F} \cdot \underset{\sim}{v} = 2\pi t$

$$\int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 2\pi t\, dt = \pi.$$

(d) Note that on $\mathcal{C}_1$, $\underset{\sim}{r}(0) = \underset{\sim}{j}$, $\underset{\sim}{r}(\pi) = -\underset{\sim}{j} + \underset{\sim}{k}$.

And on $\mathcal{C}_2$, $\underset{\sim}{r}(0) = \underset{\sim}{j}$, $\underset{\sim}{r}(1) = -\underset{\sim}{j} + \underset{\sim}{k}$.

Thus $\mathcal{C}_1$ and $\mathcal{C}_2$ are both paths from $(0,1,0)$ to $(0,-1,1)$, that is, they have the same initial point and same final point.

If $\underset{\sim}{F}$ were conservative then $\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r}$ would only depend on the endpoints of $\mathcal{C}$ so we would have $\int_{\mathcal{C}_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{\mathcal{C}_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}$.

However, in our case, we saw that

$$\int_{\mathcal{C}_1} \underset{\sim}{F} \cdot d\underset{\sim}{r} \neq \int_{\mathcal{C}_2} \underset{\sim}{F} \cdot d\underset{\sim}{r}.$$

Therefore, $\underset{\sim}{F}$ cannot be conservative.

[Alternative solution: compute $\underset{\sim}{\nabla} \times \underset{\sim}{F}$ and show that it is not $\underset{\sim}{0}$. Thus $\underset{\sim}{F}$ cannot be conservative (since $\underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{0}$ is a necessary condition for $\underset{\sim}{F}$ to be conservative).]

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & -\pi z & x - \pi x y \end{vmatrix} = \underset{\sim}{i}(-\pi x + \pi) + \underset{\sim}{j}(1 - \pi y) + \underset{\sim}{k}(0) \neq \underset{\sim}{0}$$

Exercise 4 $\quad x = u - v, \quad y = 2u + v.$

$R: \quad y = -x, \quad y = 6 - x, \quad y = 2x, \quad y = 3 + 2x.$

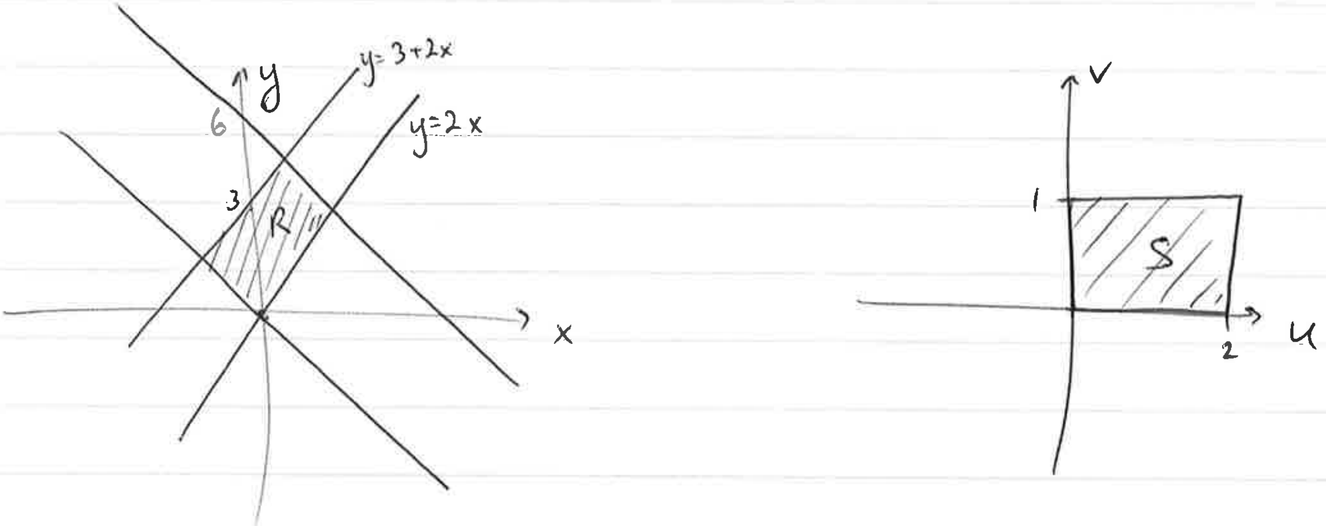
(a)

$y = -x: \quad 2u + v = v - u \Rightarrow 3u = 0 \Rightarrow u = 0$

$y = 6 - x: \quad 2u + v = 6 + v - u \Rightarrow 3u = 6 \Rightarrow u = 2$

$y = 2x: \quad 2u + v = 2u - 2v \Rightarrow 3v = 0 \Rightarrow v = 0$

$y = 3 + 2x: \quad 2u + v = 3 + 2u - 2v \Rightarrow 3v = 3 \Rightarrow v = 1$




(b) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = 1/3.$$

(c) $$\iint_R (x+y)(y-2x+1)\,dx\,dy = \iint_S (u-v+2u+v)(2u+v-2u+2v+1)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

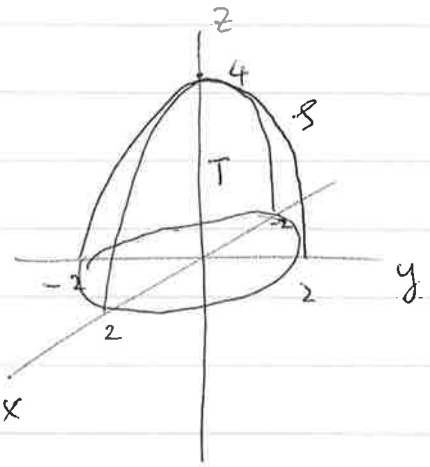
$$= \iint_S 3u\cdot(3v+1)\cdot 3\,du\,dv = 9\int_0^2 du \int_0^1 dv\ (3uv+u)$$

$$= 9\int_0^2 du\left(\tfrac{3}{2}uv^2+uv\right)\Big|_0^1 = 9\int_0^2 \tfrac{3}{2}u+u\ du = \frac{9\cdot 5}{4}u^2\Big|_0^2$$

$$= \frac{45}{4}\cdot 4 = \underline{\underline{45}}$$

**Exercise 5** $S$: $z = 4-x^2-y^2$, $z \geq 0$

$T$: region between $S$ and $xy$-plane.




(a) $T$: use cylindrical coords.

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

$z = 4-(x^2+y^2) = 4-r^2$ on $S$, so

$0 \leq z \leq 4-r^2$.

Thus $$\iiint_T z+x^2+y^2\,\underbrace{dV}_{r\,dr\,d\theta\,dz} = \int_0^{2\pi} d\theta \int_0^2 dr \int_0^{4-r^2} (z+r^2)\,r\,dz$$

$$= 2\pi\int_0^2 dr\left(\tfrac{1}{2}rz^2+r^3z\right)\Big|_{z=0}^{4-r^2} = 2\pi\int_0^2 \tfrac{1}{2}r(4-r^2)^2+r^3(4-r^2)\,dr$$

$$= 2\pi\int_0^2 \tfrac{1}{2}r\left[16-8r^2+r^4\right]+4r^3-r^5\,dr =$$

$$= 2\pi \int_0^2 8r - 4r^3 + \tfrac{1}{2}r^5 + 4r^3 - r^5 \, dr$$

$$= 2\pi \int_0^2 8r - \tfrac{1}{2}r^5 \, dr = 2\pi \left[ 4r^2 - \frac{1}{12} r^6 \right]_0^2$$

$$= 2\pi \left[ 16 - \frac{1}{12} . 64 \right] = \frac{2\pi}{3} [ 48 - 16 ] = \frac{2\pi}{3} . 32 = \frac{64\pi}{3} .$$

(b) $S : z = 4 - x^2 - y^2 = f(x,y)$, so $S$ is graph of

the function $f(x,y) = 4 - x^2 - y^2$ over the region $R : x^2 + y^2 \le 4$ in the $x$-$y$ plane.

$$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y, \quad \text{so} \quad dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx\,dy$$

$$= \sqrt{1 + 4x^2 + 4y^2} \, dx\,dy$$

Thus $\iint_S 4z - 17 \, dS = \iint_R [4(4 - x^2 - y^2) - 17] \sqrt{1 + 4x^2 + 4y^2} \, \underbrace{dx\,dy}_{r\,dr\,d\theta}$

$$= \int_0^{2\pi} d\theta \int_0^2 (16 - 4r^2 - 17) \sqrt{1 + 4r^2} \cdot r \, dr$$

$$= \int_0^{2\pi} d\theta \int_0^2 (-1 - 4r^2) \sqrt{1 + 4r^2} \, r \, dr$$

$$= -2\pi \int_0^2 (1 + 4r^2)^{3/2} r \, dr = -2\pi \cdot \frac{2}{5} \cdot \frac{1}{8} (1 + 4r^2)^{5/2} \Big|_0^2$$

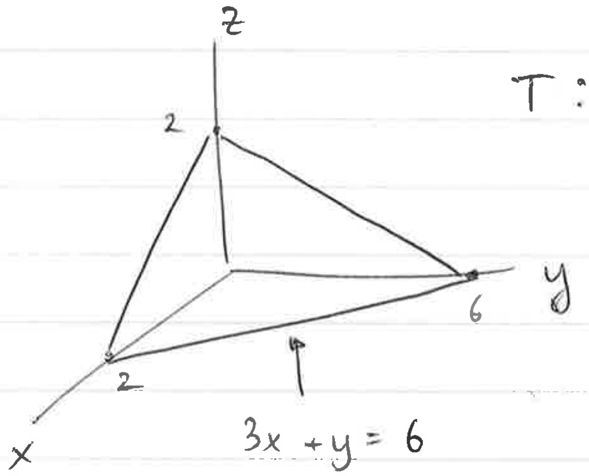
$$= -\frac{\pi}{10} \left[ (17)^{5/2} - 1 \right] = \frac{\pi}{10} \left[ 1 - 17^{5/2} \right].$$

Exercise 6 T: region under $3x+y+3z=6$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

(a) when $x=0, y=0 \Rightarrow 3z=6 \Rightarrow z=2$

$x=0, z=0 \Rightarrow y=6$

$y=0, z=0 \Rightarrow 3x=6 \Rightarrow x=2.$




T: $0 \leq x \leq 2$

$0 \leq y \leq 6-3x$

$0 \leq z \leq 2-x-y/3$

$$\iiint_T xy\, dV = \int_0^2 dx \int_0^{6-3x} dy \int_0^{2-x-y/3} x \, dz$$

$$= \int_0^2 dx \int_0^{6-3x} dy \left( x \left(2-x-y/3\right)\right)$$

$$= \int_0^2 dx \left(2xy - x^2y - \frac{xy^2}{6}\right)\Big|_0^{6-3x}$$

$$= \int_0^2 2x(6-3x) - x^2(6-3x) - \tfrac{1}{6}x(6-3x)^2 \, dx$$

$$= \int_0^2 12x - \cancel{6x^2} - 6x^2 + 3x^3 - 6x + \cancel{6x^2} - \frac{3}{2}x^3 \, dx$$

$$= \int_0^2 6x - 6x^2 + 3/2\, x^3 \, dx = 3x^2 - 2x^3 + 3/8\, x^4 \Big|_0^2$$

$$= 12 - 16 + 6 = 2$$

(b) $S: \quad 3x + y + 3z = 6, \qquad x \geq 0, \; y \geq 0, \; z \geq 0.$

Let $G(x,y,z) = 3x + y + 3z$, then $S: G(x,y,z) = 6$.

$$\nabla G = 3\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}, \qquad \frac{\partial G}{\partial z} = 3.$$

Thus $dS = \dfrac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|} \, dx\,dy = \dfrac{\sqrt{9+1+9}}{3} \, dx\,dy = \dfrac{\sqrt{19}}{3} \, dx\,dy.$

And $S$ is the level set $G(x,y,z) = 6$ over the region $R$: $0 \leq x \leq 2$, $0 \leq y \leq 6 - 3x$ in the $xy$-plane.

Thus $\displaystyle\iint_S x(3z+y)\, dS = \iint_R x(6-3x) \frac{\sqrt{19}}{3} \, dx\,dy.$

$$= \frac{\sqrt{19}}{3} \int_0^2 dx \int_0^{6-3x} 6x - 3x^2 \, dy$$

$$= \frac{\sqrt{19}}{3} \int_0^2 dx \left( 6x(6-3x) - 3x^2(6-3x) \right)$$

$$= \frac{\sqrt{19}}{3} \int_0^2 36x - 18x^2 - 18x^2 + 9x^3 \, dx$$

$$= \frac{\sqrt{19}}{3} \left[ 18x^2 - 12x^3 + \tfrac{9}{4} x^4 \right]_0^2$$

$$= \frac{\sqrt{19}}{3} \left[ 72 - 96 + 36 \right] = 4\sqrt{19}.$$

**Exercise 7**

$$\underset{\sim}{F} = \overset{F_1}{(z - x)}\underset{\sim}{i} + \overset{F_2}{(3y + z^2)}\underset{\sim}{j} + \overset{F_3}{x^2 y}\,\underset{\sim}{k}$$

(a) $$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

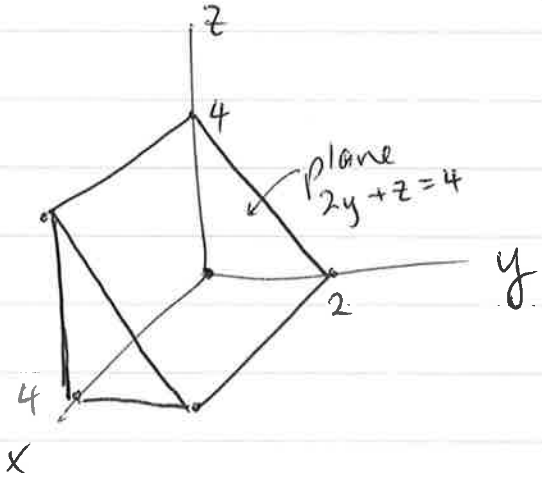
$$= -1 + 3 + 0 = 2$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z - x & 3y + z^2 & x^2 y \end{vmatrix}$$

$$= \underset{\sim}{i}(x^2 - 2z) + \underset{\sim}{j}(1 - 2xy) + \underset{\sim}{k}(0 - 0)$$

$$= \underset{\sim}{i}(x^2 - 2z) + \underset{\sim}{j}(1 - 2xy).$$

(b) $T$ bounded by $2y + z = 4$, $x = 4$, $x = 0$, $y = 0$, $z = 0$.




$$\underset{S}{\oiint} \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \underset{T}{\iiint} \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV$$

$$= 2 \underset{T}{\iiint} dV = 2 \cdot \mathrm{vol}(T)$$

$$= 2 \cdot \tfrac{1}{2} \cdot 4 \cdot 2 \cdot 4$$

$$= 32.$$

(c) Note that $\mathcal{C}$ is the boundary of the top face $\mathcal{R}$ of $T$, with the orientation induced by the upwards-pointing unit normal to the face $\mathcal{R}$.

By Stokes' theorem,

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_{\mathcal{R}} (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS$$

Now the surface $\mathcal{R}$ is defined by $2y + z = 4$, over the region $D: 0 \leq x \leq 4,\ 0 \leq y \leq 2$ in the $xy$-plane.

Let $G(x,y,z) = 2y + z$, then $\mathcal{R}: G(x,y,z) = 4$ over $D$.

$$\underset{\sim}{\nabla} G = 2\underset{\sim}{j} + \underset{\sim}{k}, \quad \frac{\partial G}{\partial z} = 1.$$

Thus $d\underset{\sim}{S} = \hat{\underset{\sim}{N}}\, dS = \pm \dfrac{\underset{\sim}{\nabla} G}{\frac{\partial G}{\partial z}}\, dx\, dy = \pm (2\underset{\sim}{j} + \underset{\sim}{k})\, dx\, dy$

To get the correct orientation, choose '+'.

$$\Rightarrow d\underset{\sim}{S} = (2\underset{\sim}{j} + \underset{\sim}{k})\, dx\, dy$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \underset{\sim}{i}(x^2 + 2z) + \underset{\sim}{j}(1 - 2xy)$$

$$\Rightarrow (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS = 2(1 - 2xy)\, dx\, dy.$$

So $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D 2(1-2xy)\,dx\,dy$

$= 2\int_0^4 dx \int_0^2 (1-2xy)\,dy$

$= 2\int_0^4 dx\,(y - xy^2)\Big|_{y=0}^{2}$

$= 2\int_0^4 2-4x\;dx = 2\left[2x - 2x^2\right]_0^4$

$= 2[8-32] = -48.$

# Exercise 8

$$\underset{\sim}{F} = (x^2+y)\underset{\sim}{i} + (yz-y-x^2)\underset{\sim}{j} + z\sqrt{x^2+y^2}\,\underset{\sim}{k}$$

(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 2x + z - 1 + \sqrt{x^2+y^2}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y & yz-y-x^2 & z\sqrt{x^2+y^2} \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\frac{zy}{\sqrt{x^2+y^2}} - y\right) + \underset{\sim}{j}\left(0 - \frac{zx}{\sqrt{x^2+y^2}}\right) + \underset{\sim}{k}\,(-2x-1)$$

$$= \underset{\sim}{i}\left(\frac{zy}{\sqrt{x^2+y^2}} - y\right) - \underset{\sim}{j}\,\frac{zx}{\sqrt{x^2+y^2}} - \underset{\sim}{k}\,(2x+1).$$

(b) $R$ : $x^2+y^2=4$, $0 \le z \le 2$

$T$ : Solid cylinder bounded by $R$, $z=0$, $z=2$.

$$\oiint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = \iiint_T 2x + z - 1 + \sqrt{x^2+y^2}\, dV$$

Use cylindrical coords; then $T$: $0 \le \theta \le 2\pi$

$0 \le r \le 2$

$0 \le z \le 2$.

$(dV = r\,dr\,d\theta\,dz.)$

We get

$$\int_0^{2\pi} d\theta \int_0^2 dr \int_0^2 (2r\cos\theta + z - 1 + r)\, r\, dz$$

$$= \int_0^{2\pi} d\theta \int_0^2 dr \int_0^2 2r^2\cos\theta + rz - r + r^2 \, dz$$

$$= \int_0^{2\pi} d\theta \int_0^2 4r^2\cos\theta + 2r - 2r + 2r^2 \, dr$$

$$= \int_0^{2\pi} \frac{32}{3}\cos\theta + 16/3 \, d\theta$$

$$= \left(\frac{32}{3}\sin\theta + \frac{16\theta}{3}\right)\Bigg|_0^{2\pi} = \frac{32\pi}{3}.$$

(c) $\mathcal{C}$: $x^2+y^2=4$, $z=2$, oriented anticlockwise (seen from above).
Let $D$: $x^2+y^2 \leq 4$, $z=2$, then if $\hat{N} = \underset{\sim}{k}$ we have that $\mathcal{C}$ is the boundary curve of $D$ with the induced orientation.

By Stokes' theorem,

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS.$$

$$(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}} = (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{k}} = -2x - 1$$

$$\Rightarrow \oint_{C} \underset{\sim}{F} \cdot d\underset{\sim}{r} = - \iint_{D} 2x+1 \; \overset{r\,dr\,d\theta}{dS} = - \int_0^{2\pi} d\theta \int_0^2 (2r\cos\theta + 1) r \, dr$$

$$= - \int_0^{2\pi} \left[ \tfrac{2}{3} r^3 \cos\theta + \tfrac{1}{2} r^2 \right]_0^2 d\theta$$

$$= - \int_0^{2\pi} \tfrac{16}{3} \cos\theta + 2 \; d\theta$$

$$= - \left[ \tfrac{16}{3} \sin\theta + 2\theta \right]_0^{2\pi} = - 4\pi .$$