THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM I: MAT300 Vector Analysis
DATE: 11. December 2018, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators permitted in accordance with TN faculty rules
THE EXERCISE SHEET CONSISTS OF 4 EXERCISES ON 3 PAGES
+ 1 PAGE WITH FORMULAS.
EACH OF THE 11 PARTS 1a, 1b, 1c, 1d, 2a, 2b, 3a, 3b, 4a, 4b, 4c ARE WORTH EQUAL MARKS.

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = \frac{\pi}{2}t^2\,\mathbf{i} + \sin(\pi t)\,\mathbf{j} + \cos(\pi t)\,\mathbf{k}\,, \quad -1 \le t \le 1$.

a) (i) Show that $\mathscr{C}$ is a closed curve.
(ii) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \frac{1}{2}$.

b) Compute the line integral

$$\int_{\mathscr{C}} \frac{1}{\sqrt{\frac{2}{\pi}x + y^2 + z^2}}\, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x, y, z) = (yz^2 + \sin z)\,\mathbf{i} + xz^2\,\mathbf{j} + (2xyz + x\cos z - 3z^2)\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.

Let $\mathbf{G}$ be another vector field, defined by the formula

$$\mathbf{G}(x, y, z) = \frac{z\,\mathbf{j} - y\,\mathbf{k}}{y^2 + z^2}\,.$$

Note that $\mathbf{G}$ is defined on the domain $D = \{(x, y, z) \in \mathbb{R}^3 : y \text{ and } z \text{ not both zero}\}$, that is, D equals all of three-dimensional space $\mathbb{R}^3$ excluding the x-axis.

d) Compute the line integral

$$\oint_{\mathscr{C}} (\mathbf{F} + \mathbf{G}) \bullet d\mathbf{r}\,.$$

Is the vector field $\mathbf{F} + \mathbf{G}$ conservative? Give a reason to justify your answer.

EXERCISE 2

Consider the transformation $u = x + y$, $v = x$, between the (x, y)-coordinates and the (u, v)-coordinates. Let R be the region in the xy-plane bounded by

$$y = 2 - x, \quad x = 2, \quad \text{and} \quad y = \frac{1}{x} - x\,.$$

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation. Make sure to clearly label all lines and curves that you draw, and clearly show the coordinates of each vertex of R and S.

b) (i) Calculate the Jacobi determinant $\frac{\partial(x,y)}{\partial(u,v)}$.

(ii) Give a brief explanation of why the double integral

$$\iint_R xe^{x^2+xy}\,dx\,dy$$

must be a positive number.

(iii) Use the change of coordinates given above to compute the double integral

$$\iint_R xe^{x^2+xy}\,dx\,dy\,.$$

EXERCISE 3

Let the surface $\mathscr{S}$ be defined by $x^2 + y^2 = 2z^2$ for $1 \le z \le \sqrt{2}$ and $x \ge 0$. Note that $\mathscr{S}$ is the *front half* $(x \ge 0)$ of the piece of a circular cone between $z = 1$ and $z = \sqrt{2}$.

a) (i) Suppose that a function $f(x, y, z)$ satisfies

$$f(x, -y, z) = -f(x, y, z) \quad \text{for all points} \quad (x, y, z) \in \mathbb{R}^3\,.$$

Carefully explain why it then follows that

$$\iint_{\mathscr{S}} f(x, y, z)\,dS = 0\,.$$

(ii) Compute the surface integral

$$\iint_{\mathscr{S}} (xy^3 z + \sin(y)e^{y^2} + 1)\,dS\,.$$

Let T be the solid region bounded by the surface $\mathscr{S}$, the two horizontal planes $z = 1$ and $z = \sqrt{2}$, and the yz-plane.

b) Compute the triple integral

$$\iiint_T xz\,dV\,.$$

EXERCISE 4

Consider the vector field $\mathbf{F}(x, y, z) = (2xy^2z)\,\mathbf{i} + x\,\mathbf{j} + (1 + xz - y^2z^2)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let T be the solid tetrahedron (triangular-based pyramid) with vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$.
Let $\mathscr{S}$ be the closed boundary surface of T, equipped with the outwards pointing unit normal vector field.

b) Use the divergence theorem to compute the flux

$$\oiint_{\mathscr{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\,.$$

Now let $\mathscr{R}$ be the triangular surface with vertices at $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$, equipped with the upwards-pointing unit normal vector field.

c) Compute the flux

$$\iint_{\mathscr{R}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\,.$$

Hint: Compute the flux through the three other faces of T, and then apply the divergence theorem together with your result from part (b).

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right|\, dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}\,:\, G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}\,:\, G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint_D \nabla\bullet\mathbf{F}\ dV = \oiint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\ dS = \oint_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.

Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad \cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

Exercise 1

$$\mathcal{C} : \underset{\sim}{r}(t) = \overset{x(t)}{\frac{\pi}{2} t^2}\, \underset{\sim}{i} + \overset{y(t)}{\sin(\pi t)}\, \underline{j} + \overset{z(t)}{\cos(\pi t)}\, \underline{k}$$

$$-1 \leq t \leq 1.$$

(a) (i) $\mathcal{C}$ is closed if its endpoints coincide.

$$\underset{\sim}{r}(-1) = \frac{\pi}{2} \underset{\sim}{i} + \sin(-\pi)\underline{j} + \cos(-\pi)\underline{k}$$

$$= \frac{\pi}{2} \underline{i} - \underline{k}$$

$$\underset{\sim}{r}(1) = \pi/2\, \underline{i} + \sin(\pi)\underline{j} + \cos(\pi)\underline{k}$$

$$= \frac{\pi}{2} \underline{i} - \underline{k}$$

Since $\underline{r}(-1) = \underline{r}(1)$, $\mathcal{C}$ is closed.

(ii)

$$\frac{d\underset{\sim}{r}}{dt}(t) = \pi t\, \underset{\sim}{i} + \pi \cos(\pi t)\, \underline{j} - \pi \sin(\pi t)\, \underline{k}$$

$$\frac{d\underline{r}}{dt}\left(\frac{1}{2}\right) = \frac{\pi}{2} \underline{i} + \pi \cos\left(\frac{\pi}{2}\right) \underline{j} - \pi \sin(\pi/2)\underline{k}$$

$$= \frac{\pi}{2} \underline{i} - \pi \underline{k}$$ is a tangent vector to $\mathcal{C}$ when $t = 1/2$, but not unit length.

$$\left| \frac{d\underline{r}}{dt}(1/2) \right| = \sqrt{\left(\frac{\pi}{2}\right)^2 + \pi^2} = \pi \sqrt{\frac{1}{4} + \frac{4}{4}} = \frac{\sqrt{5}\,\pi}{2}$$

$$\hat{\underset{\sim}{T}} = \frac{\frac{d\underline{r}}{dt}(1/2)}{\left| \frac{d\underline{r}}{dt}(1/2) \right|} = \frac{2}{\sqrt{5}\,\pi}\left(\frac{\pi}{2} \underline{i} - \pi \underline{k} \right) = \frac{1}{\sqrt{5}} \underline{i} - \frac{2}{\sqrt{5}} \underline{k}.$$

(b) $$\int_{C} \frac{1}{\sqrt{\frac{2}{\pi} x + y^2 + z^2}}\, ds = \int_{C} f(x,y,z)\, ds$$

$$ds = \left| \frac{d\underline{r}}{dt}(t) \right| dt = \sqrt{\pi^2 t^2 + \pi^2 \cos^2(\pi t) + \pi^2 \sin^2(\pi t)}$$

$$= \pi \sqrt{t^2 + 1}$$

$$f(x,y,z) = \frac{1}{\sqrt{\frac{2}{\pi} x + y^2 + z^2}} = \frac{1}{\sqrt{\frac{2}{\pi} \cdot \frac{\pi}{2} t^2 + \sin^2 \pi t + \cos^2 \pi t}}$$

$$= \frac{1}{\sqrt{t^2 + 1}}.$$

So $$\int_{C} f(x,y,z)\, ds = \int_{-1}^{1} \frac{1}{\sqrt{t^2+1}} \cdot \pi \sqrt{t^2+1}\, dt$$

$$= \pi \int_{-1}^{1} dt = 2\pi.$$

(c) $\underline{F}(x,y,z) = (yz^2 + \sin z)\underline{i} + xz^2\underline{j} + (2xyz + x\cos z - 3z^2)\underline{k}$

Try to solve $\underset{\sim}{\nabla}\phi = \underline{F} \Rightarrow$

$$\frac{\partial \phi}{\partial x} = yz^2 + \sin z \qquad (1)$$

$$\frac{\partial \phi}{\partial y} = xz^2 \qquad (2)$$

$$\frac{\partial \phi}{\partial z} = 2xyz + x\cos z - 3z^2 \qquad (3)$$

(1) $\Rightarrow \quad \phi(x,y,z) = xyz^2 + x\sin z + C_1(y,z).$

$$\frac{\partial \phi}{\partial y} = xz^2 + \frac{\partial C_1}{\partial y}(y,z) \overset{(2)}{=} xz^2 \Rightarrow \frac{\partial C_1}{\partial y}(y,z) = 0$$

$$\Rightarrow \quad C_1(y,z) = C_2(z).$$

Then $\quad \phi(x,y,z) = xyz^2 + x\sin z + C_2(z)$

$$\frac{\partial \phi}{\partial z} = 2xyz + x\cos z + \frac{\partial C_2}{\partial z}(z) \overset{(3)}{=} 2xyz + x\cos z - 3z^2$$

$$\Rightarrow \quad \frac{\partial C_2}{\partial z}(z) = -3z^2 \quad \Rightarrow \quad C_2(z) = -z^3 + C_3.$$

we may choose the final constant $C_3 = 0$, then

$$\phi(x,y,z) = xyz^2 + x\sin z - z^3.$$

(d) $\underset{\sim}{G}(x,y,z) = \dfrac{z\underline{j} - y\underline{k}}{y^2 + z^2}$.

$$\oint_{\mathcal{C}} (\underset{\sim}{F} + \underline{G}) \cdot d\underline{r} = \oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} + \oint_{\mathcal{C}} \underline{G} \cdot d\underline{r}$$

Now $\underline{F}$ is conservative, so $\displaystyle\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} = \oint_{\mathcal{C}} \underset{\sim}{\nabla}\phi \cdot d\underline{r} = \phi(\text{final}) - \phi(\text{initial})$

But since $\mathcal{C}$ is closed, the final and initial points agree, and $\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} = 0$. (Integral of a conservative vector field around a closed loop is zero.)

What about $\underset{\sim}{G}$? $\underset{\sim}{G}(x(t), y(t), z(t)) = \dfrac{\cos(\pi t)\underline{j} - \sin(\pi t)\underline{k}}{\sin^2(\pi t) + \cos^2(\pi t)}$

$$= \cos(\pi t)\underline{j} - \sin(\pi t)\underline{k}$$

$\dfrac{d\underline{r}}{dt}(t) = \pi t\,\underline{i} + \pi\cos(\pi t)\underline{j} - \pi\sin(\pi t)\underline{k}$ (from (a)).

So $\displaystyle\oint_{\mathcal{C}} \underset{\sim}{G} \cdot d\underline{r} = \oint_{-1}^{1} \underset{\sim}{G} \cdot \frac{d\underline{r}}{dt}\, dt = \int_{-1}^{1} \pi\cos^2(\pi t) + \pi\sin^2(\pi t)\, dt$

$$= 2\pi.$$

Thus $\displaystyle\oint_{\mathcal{C}} (\underset{\sim}{F} + \underset{\sim}{G}) \cdot d\underline{r} = 0 + 2\pi = \underline{\underline{2\pi}}$.

$\boxed{\text{Since the integral of } \underline{F} + \underline{G} \text{ around a closed loop was not zero, } \underline{F} + \underline{G} \text{ is } \underline{\text{NOT}} \text{ conservative}}$

Exercise 2

$u = x + y, \quad v = x.$

$R: \quad y = 2 - x \; (1), \quad x = 2 \; (2), \quad y = 1/x - x \; (3)$

(a) Find intersections in xy-plane:

$(1) \cap (2)$: $y = 2 - x \overset{(2)}{=} 2 - 2 = 0$, $x = 2$

$(1) \cap (3)$: $2 - x = y = 1/x - x$

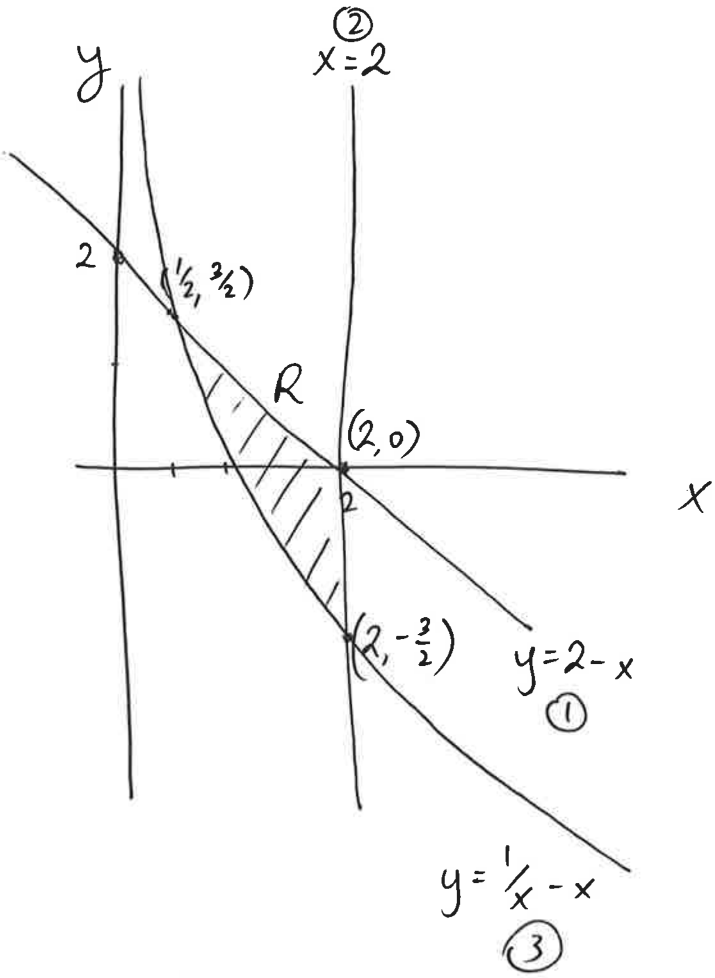
$\Rightarrow 2 = 1/x \Rightarrow x = \frac{1}{2}$

$\Rightarrow y = \frac{3}{2}$

$(2) \cap (3)$: $y = 1/x - x \overset{(2)}{=} \frac{1}{2} - 2$

$= -\frac{3}{2}$

$x = 2.$



In uv-plane:

$y = 2 - x \Rightarrow x + y = 2 \Rightarrow u = 2$

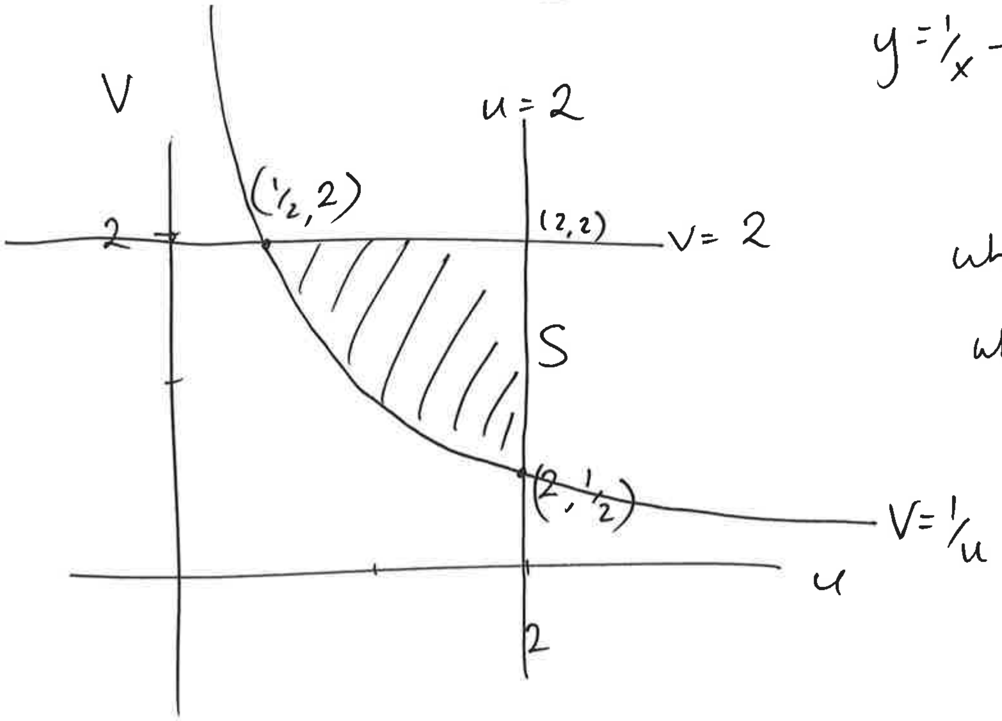
$x = 2 \Rightarrow v = 2$

$y = 1/x - x \Rightarrow y + x = 1/x \Rightarrow u = \frac{1}{v}$

$\Rightarrow v = 1/u.$

when $u = 2$, $v = \frac{1}{2}$

when $v = 2$, $u = \frac{1}{2}$.



(b) (i) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{-1} = -1.$$

(ii) $\iint_R x e^{x^2+xy}\, dx\, dy$. On the region R, the x-coordinate is always positive. Also e^{x^2+xy} is always a positive number. So the function $x e^{x^2+xy}$ is positive at each point of R. Thus, the integral of $x e^{x^2+xy}$ over R is also positive.

(iii) $$\iint_R x e^{x^2+xy}\, dx\, dy = \iint_S v e^{vu} \cdot \left| \overset{-1}{\frac{\partial(x,y)}{\partial(u,v)}} \right| du\, dv$$

$$(x^2+xy = x(x+y) = vu) \qquad = \iint_S v e^{vu}\, du\, dv.$$

$S: \ ½ \le u \le 2, \ ^1/_u \le v \le 2.$

$$\iint_S ve^{vu}\,du\,dv = \int_{1/2}^{2} du \int_{1/u}^{2} ve^{vu}\,dv.$$

The integral over v can be done by parts, but maybe it's easier if we change the order of integration?

$S: \ ½ \le v \le 2, \ ^1/_v \le u \le 2.$

$$\iint_S ve^{vu}\,du\,dv = \int_{1/2}^{2} dv \int_{1/v}^{2} ve^{vu}\,du = \int_{1/2}^{2} e^{vu}\Big|_{u=1/v}^{2}\,dv$$

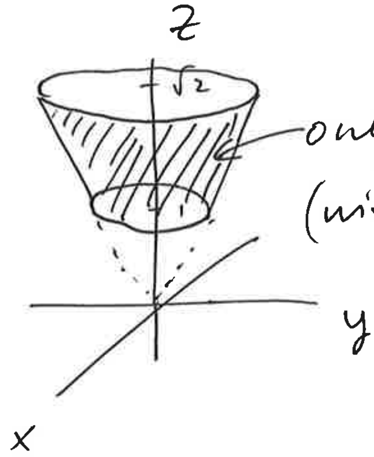
$$= \int_{1/2}^{2} \left(e^{2v} - e^{v \cdot 1/v}\right) dv = \int_{1/2}^{2} (e^{2v} - e)\,dv$$

$$= \left[\frac{1}{2}e^{2v} - ev\right]_{1/2}^{2} = \frac{1}{2}e^{4} - 2e - \left[\frac{1}{2}e^{1} - \frac{1}{2}e\right]$$

$$= \underline{\underline{\frac{1}{2}e^{4} - 2e}}.$$

Exercise 3

$S: \; x^2 + y^2 = 2z^2, \quad 1 \le z \le \sqrt{2}, \quad x \ge 0.$



only front half (with $x \ge 0$).

(a) (i) Suppose $f(x,-y,z) = -f(x,y,z)$, for all $(x,y,z) \in \mathbb{R}^3$.

So f is <u>odd</u> under the transformation

$(x,y,z) \longmapsto (x,-y,z)$. This is a reflection in the xz-plane.

But the surface S is <u>symmetric</u> under this reflection.

Thus the integral of f over the two halves of S ($y \ge 0$ and $y \le 0$) cancel each other precisely, and

$$\iint_S f(x,y,z)\, dS = 0.$$

(a) (ii) $\iint_S \left(x y^3 z + \sin(y) e^{y^2} + 1\right) dS \qquad (*)$

Let $f(x,y,z) = x y^3 z + \sin(y)\, e^{y^2}$.

Then $f(x,-y,z) = x(-y)^3 z + \sin(-y)\, e^{(-y)^2}$

$$= -xy^3 z - \sin y\, e^{y^2}$$

$$= -f(x,y,z).$$

So $\iint_S f(x,y,z)\, dS = 0$ by (a)(i). Thus

$$(*) = \iint_S 1 \cdot dS = \text{area}(S).$$

Now $S: x^2 + y^2 = 2z^2 \Rightarrow x^2 + y^2 - 2z^2 = 0$.

Let $G(x,y,z) = x^2 + y^2 - 2z^2$, then $S: G(x,y,z) = 0$.

$\underset{\sim}{\nabla} G = 2x + 2y - 4z, \qquad \dfrac{\partial G}{\partial z} = -4z.$

$$dS = \frac{|\underline{\nabla} G|}{\left|\frac{\partial G}{\partial z}\right|}\, dx\, dy = \frac{\sqrt{4x^2 + 4y^2 + 16z^2}}{4z}\, dx\, dy$$

$$= \frac{2\sqrt{x^2 + y^2 + 4z^2}}{4z}\, dx\, dy = \frac{\sqrt{2z^2 + 4z^2}}{2z}\, dx\, dy$$

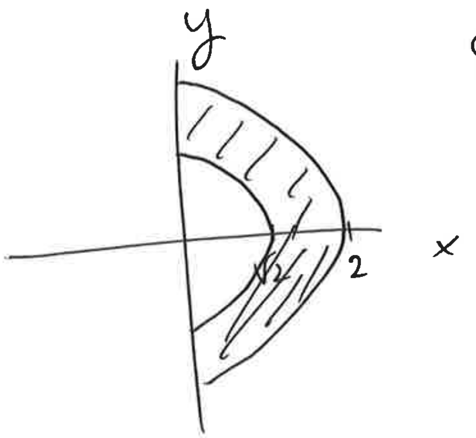
$$= \frac{\sqrt{6}}{2}\, dx\, dy.$$

Projection of S onto xy-plane:

when $z = 1$: $x^2 + y^2 = 2z^2 = 2$ circle radius $\sqrt{2}$

$z = \sqrt{2}$: $x^2 + y^2 = 2z^2 = 4$ circle radius 2.

also $x \geq 0$.



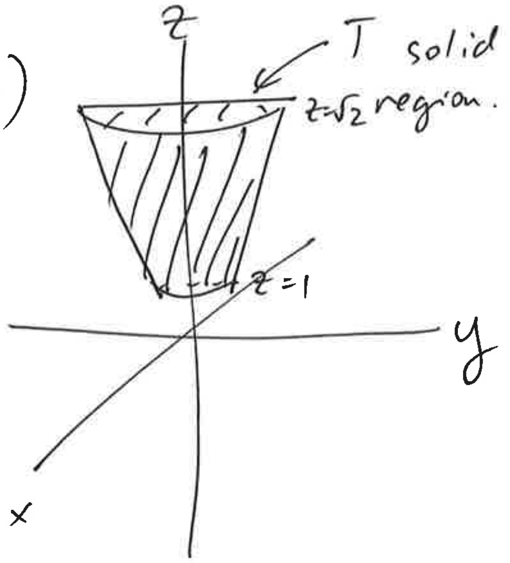
So $R \subseteq \mathbb{R}^2$ is defined by:

$$R: \quad -\pi/2 \leq \theta \leq \pi/2$$
$$\sqrt{2} \leq r \leq 2.$$

$$(*) = \iint_S 1\, dS = \int_{-\pi/2}^{\pi/2} d\theta \int_{\sqrt{2}}^{2} \frac{\sqrt{6}}{2}\, r\, dr$$

$$= \pi \cdot \frac{\sqrt{6}}{2} \cdot \frac{1}{2} r^2 \Big|_{\sqrt{2}}^{2} = \frac{\sqrt{6}\,\pi}{4}(4 - 2) = \frac{\sqrt{6}\,\pi}{2}.$$

(b)



Cylindrical coordinates:

$$T: \quad -\pi/2 \le \theta \le \pi/2$$
$$1 \le z \le \sqrt{2}$$
$$0 \le r \le \sqrt{2}\, z.$$

$$x^2+y^2 = 2z^2 \Rightarrow r^2 = 2z^2 \Rightarrow r = \sqrt{2}\, z$$

$$\iiint_T xz\, dV = \int_{-\pi/2}^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2} z} r\cos\theta\, z\, r\, dr\, dz\, d\theta$$

$x = r\cos\theta$

$$= \int_{-\pi/2}^{\pi/2} \int_1^{\sqrt{2}} \frac{1}{3} r^3 z\cos\theta \Big|_0^{\sqrt{2} z} dz\, d\theta = \int_{-\pi/2}^{\pi/2} \int_1^{\sqrt{2}} \frac{1}{3} z\cos\theta \big(\underbrace{2\sqrt{2}\, z^3}_{z^4}\big)\, dz\, d\theta$$

$$= \frac{2\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \frac{1}{5} z^5 \cos\theta \Big|_1^{\sqrt{2}} d\theta = \frac{2\sqrt{2}}{15} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\sqrt{2}-1)\cos\theta\, d\theta$$

$$= \frac{2\sqrt{2}(4\sqrt{2}-1)}{15} \cdot \underbrace{\sin\theta \Big|_{-\pi/2}^{\pi/2}}_{2} = \frac{4\sqrt{2}(4\sqrt{2}-1)}{15}.$$

Exercise 4

$$\underline{F}(x,y,z) = (2xy^2z)\underline{i} + x\underline{j} + (1 + xz - y^2z^2)\underline{k}$$

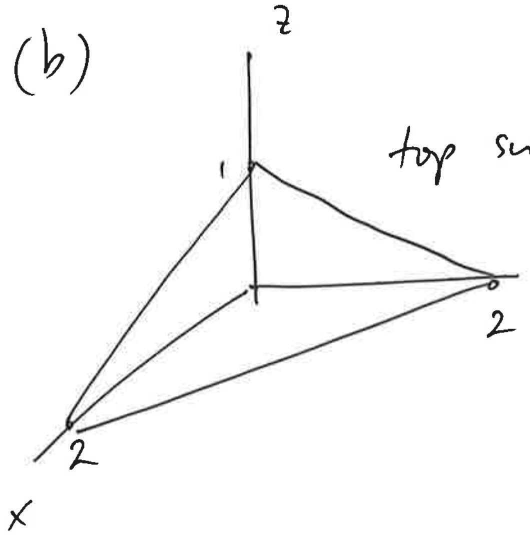
(a) $\underset{\sim}{\nabla} \cdot \underline{F} = 2y^2z + 0 + x - 2y^2z$

$= x.$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & j & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2z & x & 1 + xz - y^2z^2 \end{vmatrix}$$

$$= \underset{\sim}{i}\left(-2yz^2\right) + j\left(2xy^2 - z\right) + \underline{k}\left(1 - 4xyz\right).$$

(b)



top surface: $\frac{x}{2} + \frac{y}{2} + z = 1$

$x + y + 2z = 2.$

So T: $0 \le x \le 2$

$0 \le y \le 2 - x$

$0 \le z \le 1 - \frac{x}{2} - \frac{y}{2}$.

S is boundary surface of T with outward unit normal vector field.

$$\oiint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_T \underline{\nabla} \cdot \underline{F}\, dV$$ by divergence theorem.

$$= \int_0^2 \int_0^{2-x} \int_0^{1-\frac{x}{2}-\frac{y}{2}} x \, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} x\left(1 - \frac{x}{2} - \frac{y}{2}\right) dy\, dx$$

$$= \int_0^2 \int_0^{2-x} x - \frac{x^2}{2} - \frac{xy}{2} \, dy\, dx$$

$$= \int_0^2 xy - \frac{x^2 y}{2} - \frac{xy^2}{4} \Big|_0^{2-x} dx$$

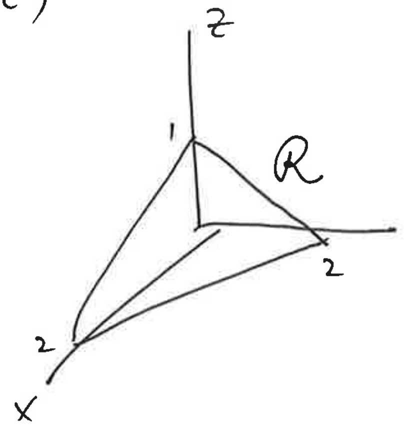
$$= \int_0^2 x(2-x) - \frac{x^2(2-x)}{2} - \frac{x\overbrace{(2-x)^2}^{4-4x+x^2}}{4} \, dx$$

$$= \int_0^2 2x - x^2 - x^2 + x^3/2 - x + x^2 - \frac{x^3}{4} \, dx$$

$$= \int_0^2 x - x^2 + \tfrac{1}{4} x^3 \, dx = \frac{x^2}{2} - x^3/3 + \frac{1}{16} x^4 \Big|_0^2$$

$$= 2 - 8/3 + 1 = 3 - 8/3 = 9/3 - 8/3 = 1/3.$$

(c)



R: $x+y+2z=2$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

upwards pointing unit normal vector field.

Let $\mathcal{A}$ be the part of $\mathcal{S}$ in the xy-plane, $\mathcal{B}$ be the part of $\mathcal{S}$ in the yz-plane, $\mathcal{D}$ be the part of $\mathcal{S}$ in the xz-plane, all with outward unit normal vector field.

Then $\oiint_{\mathcal{S}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS$

$$= \left(\iint_{R} + \iint_{\mathcal{A}} + \iint_{\mathcal{B}} + \iint_{\mathcal{D}} \right) \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS.$$

From (b), $\oiint_{\mathcal{S}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = 1/3$.

On $\mathcal{A}$: $\hat{\underset{\sim}{N}} = -\underset{\sim}{k}$, and $z=0$, so $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -(1+xz-y^2z^2)$
$= -1$.

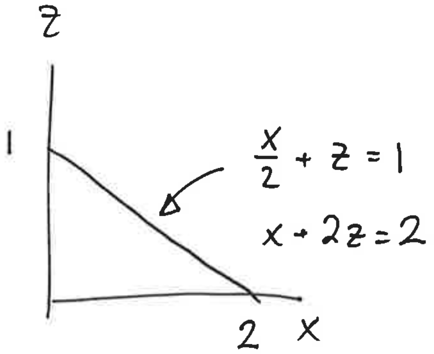
Thus $\iint_{\mathcal{A}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = \iint_{\mathcal{A}} -1\, dS = -\text{area}(\mathcal{A}) = -\frac{1}{2} \cdot 2 \cdot 2$
$= -2$.

On $\mathcal{B}$: $\hat{\underset{\sim}{N}} = -\underset{\sim}{i}$ and $x=0$, so $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -2xy^2z = 0$.

So $\iint_{\mathcal{B}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = 0$.

On ①, $\hat{\underset{\sim}{N}} = -\underset{\sim}{j}$ and $y = 0$, so $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -x$.

$$\iint_{D} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = -\iint_{D} x\, dS = -\int_0^2 \int_0^{1 - x/2} x \, dz\, dx$$



D: $0 \le x \le 2$
$0 \le z \le 1 - x/2$

$$= -\int_0^2 \cancel{z}\, x(1 - x/2)\, dx$$

$$= -\int_0^2 x - \frac{x^2}{2}\, dx$$

$$= -\left(\frac{x^2}{2} - \frac{x^3}{6}\right)_0^2$$

$$= -\left(2 - 8/6\right) = -2/3.$$

So
$$\iint_{R} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = \left(\oiint_{S} - \iint_{A} - \iint_{B} - \iint_{D}\right) \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS$$

$$= 1/3 + 2 - 0 + 2/3 = \underline{\underline{3}}$$