THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM I: MAT300 Vector Analysis (Continuation Exam)
DATE: Monday 5. March 2018, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators permitted in accordance with TN faculty rules
THE EXERCISE SHEET CONSISTS OF 4 EXERCISES ON 2 PAGES
+ 1 PAGE WITH FORMULAS.
EACH OF THE 11 PARTS 1a, 1b, 1c, 1d, 2a, 2b, 2c, 3a, 3b, 4a, 4b ARE WORTH EQUAL MARKS.

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = \sqrt{2}\sin(2t)\,\mathbf{i} - \mathbf{j} + \sqrt{2}\cos(2t)\,\mathbf{k}\,, \quad 0 \le t \le \pi$.

a) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \pi/8$.

b) Compute the line integral

$$\int_{\mathscr{C}} 2z^2 - x^2 y\, ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x, y, z) = \frac{z\,\mathbf{i} + y\,\mathbf{j} - x\,\mathbf{k}}{x^2 + z^2}\,.$$

c) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

d) Show that $\mathscr{C}$ is a closed curve, and then, using your answer to part *(c)*, state whether $\mathbf{F}$ is a conservative vector field. Give a reason to justify your answer.

EXERCISE 2

Consider the transformation $u = y^2 - x$, $v = x + y$, between the (x, y)-coordinates and the (u, v)-coordinates.
Let R be the bounded region in the upper half of the xy-plane between the lines
$x = y^2 - 1$, $x = y^2 + 1$, $x + y = 2$, and the x-axis.

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation.

b) Find the Jacobi determinants

$$\frac{\partial(x, y)}{\partial(u, v)} \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)}\,.$$

c) Use the change of coordinates given above to compute the double integral

$$\iint_R (2y + 1)(x + y + 2)\, dA\,.$$

EXERCISE 3

Let the surface $\mathscr{S}$ be defined by $z = 2 - x^2 - y^2$ for $0 \leq z \leq 1$. Note that $\mathscr{S}$ is an inverted paraboloid whose top has been "cut off".

Let T be the solid region bounded by the surface $\mathscr{S}$ and the two horizontal planes $z = 0$ and $z = 1$.

a) Compute the triple integral

$$\iiint_T z\, dV\,.$$

b) Compute the area of $\mathscr{S}$, given by

$$\iint_{\mathscr{S}} dS\,.$$

EXERCISE 4

Consider the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + y^2\,\mathbf{j} + (1 - xz - y^2)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{C}$ be the closed triangular loop with corners at the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The orientation on $\mathscr{C}$ is anticlockwise when viewed from the origin.

b) Use Stokes' theorem to compute the curve integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \; dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f \;dS = \iint_R f\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} f \;dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S} : z = g(x,y)$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet \pm(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}$: $G(x,y,z) = c$, parametrised by $(x,y) \in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla\bullet\mathbf{F}\;dV = \oiint\limits_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\;dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\;dS = \oint\limits_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.
Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.
Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

1. $C: \underset{\sim}{r}(t) = \sqrt{2}\sin(2t)\,\underline{i} - \underline{j} + \sqrt{2}\cos(2t)\underline{k}, \quad 0 \le t \le \pi.$

(a) $\underset{\sim}{r}'(t) = 2\sqrt{2}\cos(2t)\,\underline{i} - 2\sqrt{2}\sin(2t)\underline{k}, \quad 0 \le t \le \pi.$

$$\underline{r}'\left(\tfrac{\pi}{8}\right) = 2\sqrt{2}\cos(\pi/4)\,\underline{i} - 2\sqrt{2}\sin(\pi/4)\,\underline{k}$$

$$= 2\sqrt{2}\cdot\tfrac{1}{\sqrt{2}}\,\underline{i} - 2\sqrt{2}\cdot\frac{1}{\sqrt{2}}\underline{k} = 2\underline{i} - 2\underline{k}$$

$$|\underline{r}'(\pi/8)| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

$$\Rightarrow \hat{\underset{\sim}{T}} = \frac{\underline{r}'(\pi/8)}{|\underline{r}'(\pi/8)|} = \tfrac{1}{\sqrt{2}}\underline{i} - \tfrac{1}{\sqrt{2}}\underline{k}$$

(b) $$\int_C 2z^2 - x^2 y\, ds = \int_0^\pi f(\underline{r}(t))\,|\underline{r}'(t)|\,dt$$

$$|\underline{r}'(t)| = \left(4\cdot 2\cdot\cos^2(2t) + 4\cdot 2\sin^2(2t)\right)^{1/2} = \sqrt{8} = 2\sqrt{2}.$$

$$f(x,y,z) = 2z^2 - x^2 y = 2\cdot 2\cos^2(2t) - 2\sin^2(2t)(-1)$$

$$= 4\cos^2(2t) + 2\sin^2(2t) = 2 + 2\cos^2(2t)$$

$$= 2 + 2\cdot\tfrac{1}{2}\cdot(1 + \cos 4t) = 3 + \cos 4t.$$

So $$\int_C f\,ds = \int_0^\pi [3 + \cos(4t)]\cdot 2\sqrt{2}\,dt = 2\sqrt{2}\left[3t + \tfrac{1}{4}\sin(4t)\right]_0^\pi$$

$$= 2\sqrt{2}\,[3\pi] = 6\sqrt{2}\,\pi.$$

(c) $\underline{F}(x,y,z) = \dfrac{z\,\underline{i} + y\,\underline{j} - x\,\underline{k}}{x^2 + z^2}$

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} = \int_0^{\pi} \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$\underline{F}(\underline{r}(t)) = \frac{\sqrt{2}\cos(2t)\,\underline{i} - \underline{j} - \sqrt{2}\sin(2t)\,\underline{k}}{2}$$

$$\underset{\sim}{F} \cdot \underline{r}'(t) = \frac{4\cos^2(2t) + 4\sin^2(2t)}{2} = 4/2 = 2.$$

So $\displaystyle\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_0^{\pi} 2\, dt = \underline{\underline{2\pi}}.$

(d) We notice that $\underset{\sim}{r}(0) = (0,-1,\sqrt{2}) = \underset{\sim}{r}(\pi)$, so $\mathcal{C}$ is a closed loop. By (c), $\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} \neq 0$.

If $\underline{F}$ were conservative, the circulation around every closed loop would be zero (by a theorem from lectures). Thus, since we have found a loop ($\mathcal{C}$) around which the circulation is <u>not</u> zero, $\underset{\sim}{F}$ cannot be conservative.

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2. $u = y^2 - x \qquad v = x + y$

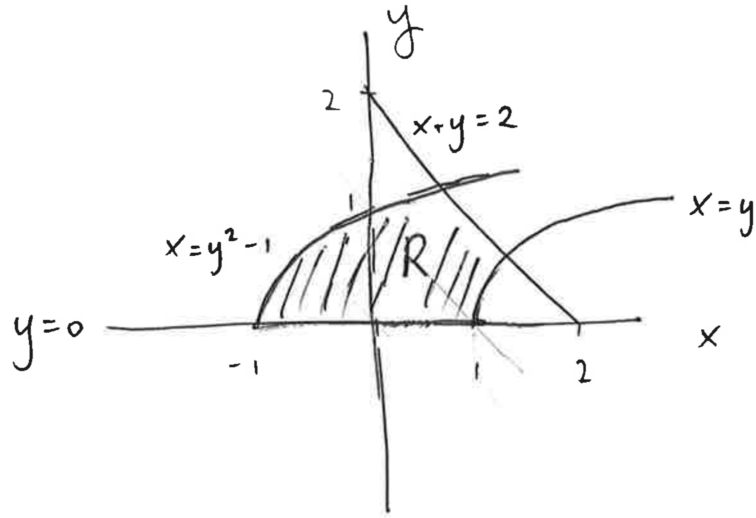
R: region between $x = y^2 - 1$, $x = y^2 + 1$, $x + y = 2$, $y = 0$ (x-axis

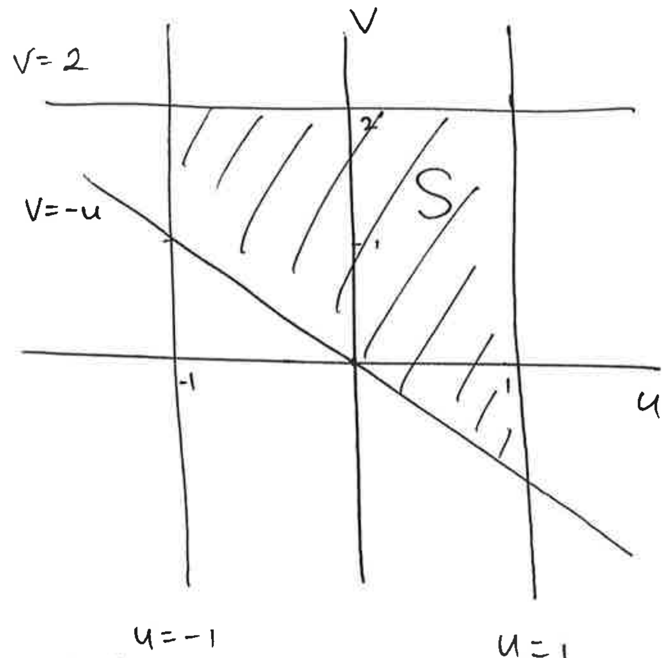
(a) $x = y^2 - 1 \Rightarrow y^2 - x = 1 \Rightarrow u = 1$

$x = y^2 + 1 \Rightarrow y^2 - x = -1 \Rightarrow u = -1$

$x + y = 2 \Rightarrow v = 2$

$y = 0 \Rightarrow u + v = y^2 - x + x + y = y^2 + y = 0 \Rightarrow v = -u.$





(b) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 2y \\ 1 & 1 \end{vmatrix} = -1 - 2y = -(1+2y)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{-1-2y} = -\frac{1}{1+2y}.$$

(c) $I = \iint_R (2y+1)(x+y+2)\,dA.$

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{1+2y}\,du\,dv$$ (since $1+2y \geq 0$ on R).

So $$I = \iint_S (2y+1)(x+y+2)\cdot\frac{1}{1+2y}\,du\,dv$$

$V = x+y$

$$= \iint_S (x+y+2)\,du\,dv = \iint_S (v+2)\,du\,dv.$$

Now $S: -1 \leq u \leq 1, \quad -u \leq v \leq 2.$

$$\Rightarrow \quad I = \int_{-1}^{1} du \int_{-u}^{2} (v+2)\,dv = \int_{-1}^{1} \left.\tfrac{1}{2}v^2 + 2v\right|_{-u}^{2} du$$

$$= \int_{-1}^{1} 2 + 4 - \left(\tfrac{1}{2}u^2 - 2u\right) du = \int_{-1}^{1} 6 + 2u - \tfrac{1}{2}u^2\,du$$

$$= \left. 6u + u^2 - \tfrac{1}{6}u^3 \right|_{-1}^{1} = \left(6+1-\tfrac{1}{6}\right) - \left(-6+1+\tfrac{1}{6}\right)$$

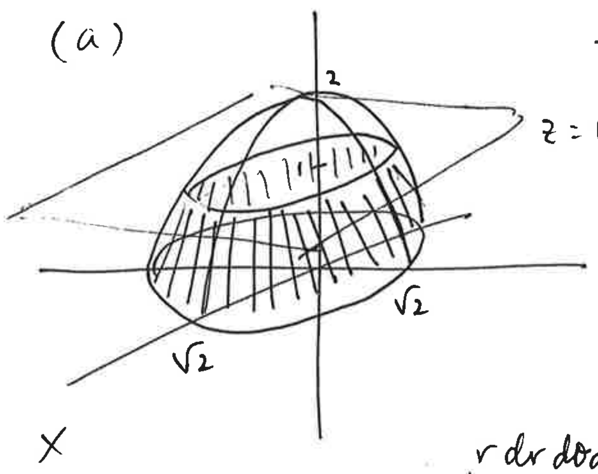
$$= 12 - \tfrac{1}{3} = \frac{35}{3}.$$

3. $S: \; z = 2 - x^2 - y^2 \;, \quad 0 \le z \le 1$

$= 2 - r^2$

T : region bounded by S and planes $z=0$, $z=1$.

(a)



$T: \quad 0 \le \theta \le 2\pi$

$0 \le z \le 1$

$0 \le r \le \sqrt{2-z}$

Let $f(x,y,z) = z$.

$$\iiint_T z \underbrace{dV}_{r\,dr\,d\theta\,dz} = \int_0^{2\pi} d\theta \int_0^1 dz \int_0^{\sqrt{2-z}} z\, r\, dr$$

$$= 2\pi \int_0^1 \frac{1}{2} z r^2 \Big|_{r=0}^{\sqrt{2-z}} dz$$

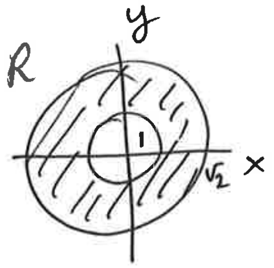
$$= 2\pi \int_0^1 \frac{1}{2} z(2-z)\, dz = 2\pi \left[\frac{1}{2} z^2 - \frac{1}{6} z^3 \right]_0^1 = \underline{\underline{\frac{2}{3}\pi}}$$

(b) $S: \; z = 2 - x^2 - y^2 = g(x,y)$

$$\Rightarrow \quad dS = \sqrt{(2x)^2 + (2y)^2 + 1}\; dx\,dy = \sqrt{4r^2+1}\; dx\,dy$$

projection of S onto xy-plane is a ring, inner radius : $z = 1 = 2 - r^2 \Rightarrow r^2 = 1 \Rightarrow r = 1$

outer radius: $z = 0 = 2 - r^2 \Rightarrow r = \sqrt{2}$.



$R: \; 0 \le \theta \le 2\pi$
$\quad\; 1 \le r \le \sqrt{2}$

Thus
$$\text{area}(S) = \iint_S 1 \cdot dS = \iint_R \sqrt{4r^2+1} \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^{\sqrt{2}} r\,(4r^2+1)^{1/2}\,dr$$

$$= 2\pi \cdot \frac{2}{3} \cdot \frac{1}{8} (4r^2+1)^{3/2} \Big|_1^{\sqrt{2}} = \frac{\pi}{6}\left[9^{3/2} - 5^{3/2}\right]$$

$$= \underline{\pi/6 \left[27 - 5^{3/2}\right]} \approx 8.283$$

4. $\underline{F}(x,y,z) = xz\,\underline{i} + y^2\underline{j} + (1 - xz - y^2)\underline{k}$

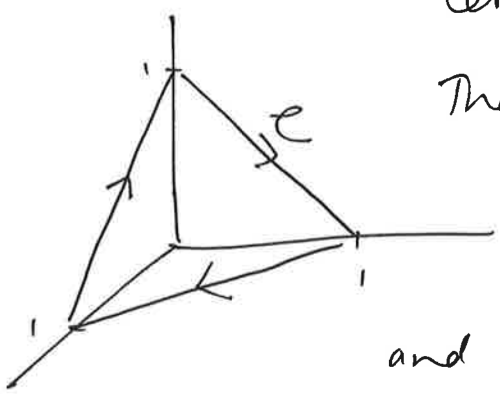
(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = z + 2y - x$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & y^2 & 1-xz-y^2 \end{vmatrix}$$

$$= \underset{\sim}{i}(-2y) + \underline{j}(x+z) + \underline{k}\cdot 0$$

$$= -2y\underline{i} + (x+z)\underline{j}$$

(b)



Let S be triangle $x+y+z=1$, $x\geq 0, y\geq 0, z\geq 0$.

Then C is the boundary curve of S.

$$\underset{\sim}{\hat{N}}\, dS = \pm\frac{(\underset{\sim}{i} + \underline{j} + \underset{\sim}{k})}{1}\, dx\,dy$$

and we take "$-$" sign to get the correct orientation on C.

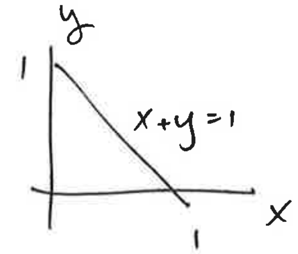
So $\underset{\sim}{\hat{N}}\, dS = (-\underline{i} - \underline{j} - \underline{k})\,dx\,dy$.

$I = \oint_C \underset{\sim}{F} \cdot d\underline{r} = \iint_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS$ by Stokes' theorem.

$$(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS = (2y - x - z)\,dx\,dy.$$

On S, $z = 1 - x - y \Rightarrow (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{N}}\, dS = (3y - 1)\,dx\,dy.$

Projection of S onto xy-plane is triangle



$$0 \le x \le 1, \quad 0 \le y \le 1 - x$$

Thus $I = \int_0^1 dx \int_0^{1-x} (3y-1)\, dy$

$$= \int_0^1 \frac{3}{2}(1-x)^2 - (1-x)\, dx = \int_0^1 \frac{3}{2} - 3x + \tfrac{3}{2}x^2 - 1 + x\, dx$$

$$= \int_0^1 \tfrac{3}{2}x^2 - 2x + \tfrac{1}{2}\, dx = \tfrac{1}{2} - 1 + \tfrac{1}{2} = 0.$$

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