THE UNIVERSITY OF STAVANGER
FACULTY OF SCIENCE AND TECHNOLOGY
EXAM: MAT300 Vector Analysis (re-sit exam)
DATE: 4. March 2019, 09:00 – 13:00
PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators permitted in accordance with TN faculty rules
THE EXERCISE SHEET CONSISTS OF 4 EXERCISES ON 3 PAGES
+ 1 PAGE WITH FORMULAS.
EACH OF THE 10 PARTS 1a, 1b, 1c, 1d, 2a, 2b, 3a, 3b, 4a, 4b ARE WORTH EQUAL MARKS.

EXERCISE 1

Consider the curve $\mathscr{C}$: $\mathbf{r}(t) = \cos(3t)\,\mathbf{i} + e^t\,\mathbf{j} - \sin(3t)\,\mathbf{k}\,,\quad 0 \le t \le 2\pi$.

a) (i) Is $\mathscr{C}$ a closed curve? Give a reason for your answer.
(ii) Find a unit tangent vector to $\mathscr{C}$ at the point corresponding to $t = \pi$.

b) Compute the line integral

$$\int_{\mathscr{C}} x^2 + y^2 + z^2 - 1\,ds\,.$$

Consider the vector field given by

$$\mathbf{F}(x, y, z) = (\sin y + ze^x)\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} + (-y\sin z + e^x)\,\mathbf{k}\,.$$

c) Show that $\mathbf{F}$ is conservative by finding a scalar potential ϕ for $\mathbf{F}$.

d) Compute the line integral

$$\int_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

EXERCISE 2

Consider the transformation $u = y$, $v = x^2 + y$, between the (x, y)-coordinates and the (u, v)-coordinates. Let R be the region in the *right half* of the xy-plane that is bounded by

$$y = 4 - x^2, \quad x = 1, \quad \text{and the } x\text{-axis}.$$

a) Sketch the given region R in the xy-plane and the region S in the uv-plane that corresponds to R under this coordinate transformation. Make sure to clearly label all lines and curves that you draw in both the xy-plane and the uv-plane.

b) (i) Give a brief explanation of why the double integral

$$\iint_R xy(x^2 + y)\, dx\, dy$$

must be a positive number.

(ii) Calculate the Jacobi determinant $\frac{\partial(x,y)}{\partial(u,v)}$.

(iii) Use the change of coordinates given above to compute the double integral

$$\iint_R xy(x^2 + y)\, dx\, dy\,.$$

EXERCISE 3

Let the surface $\mathscr{S}$ be defined by $x^2 + y^2 + z^2 = 4$ for $0 \leq z \leq 1$. Note that $\mathscr{S}$ is the part of the surface of a sphere of radius 2 that lies between the horizontal planes $z = 0$ and $z = 1$.

Let the surface $\mathscr{R}$ be the half of the surface $\mathscr{S}$ that satisfies $y \geq 0$.

a) (i) Suppose that a function $f(x, y, z)$ satisfies

$$f(x, -y, z) = f(x, y, z) \quad \text{for all points} \quad (x, y, z) \in \mathbb{R}^3\,.$$

Carefully explain why it then follows that

$$\iint_{\mathscr{S}} f(x, y, z)\, dS = 2 \iint_{\mathscr{R}} f(x, y, z)\, dS\,.$$

(ii) Compute the surface integral

$$\iint_{\mathscr{S}} x^2 z\, dS\,.$$

Let T be the solid region bounded by the surface $\mathscr{S}$, the horizontal plane $z = 0$, and the cone $x^2 + y^2 = 3z^2$.

b) Compute the volume of T, given by the triple integral

$$\iiint_T dV\,.$$

EXERCISE 4

Consider the vector field $\mathbf{F}(x, y, z) = (y - xz^2)\,\mathbf{i} + (\sin y + \frac{1}{2}x^2 z)\,\mathbf{j} + (\frac{1}{2}x^2 y + e^x)\,\mathbf{k}$.

a) Compute $\nabla \bullet \mathbf{F}$ (the divergence of $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (the curl of $\mathbf{F}$).

Let $\mathscr{C}$ be the triangular curve with vertices at $(2, 0, 0), (0, 0, 2)$, and $(0, 2, 2)$, equipped with the clockwise orientation when viewed from above.

b) Use Stokes' theorem to compute the curve integral

$$\oint_{\mathscr{C}} \mathbf{F} \bullet d\mathbf{r}\,.$$

END OF EXAM

Formulas:

Change of variables for double integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,.$$

Line integral of a function f along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} f ds = \int_a^b f(\mathbf{r}(t)) \left|\frac{d\mathbf{r}}{dt}\right| \, dt\,.$$

Line integral of a vector field $\mathbf{F} = F_1\,\mathbf{i}+F_2\,\mathbf{j}+F_3\,\mathbf{k}$, along a curve $\mathscr{C}$: $\mathbf{r}=\mathbf{r}(t)$, $a \le t \le b$:

$$\int_{\mathscr{C}} \mathbf{F}\bullet\hat{\mathbf{T}} ds = \int_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r} = \int_{\mathscr{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \mathbf{F}(\mathbf{r}(t))\bullet\frac{d\mathbf{r}}{dt}\,dt = \int_a^b (F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt})dt\,.$$

Integral of a function f over a surface $\mathscr{S}: z = g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dx\,dy\,.$$

Integral of a function f over a surface $\mathscr{S}: G(x,y,z)=c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} f\ dS = \iint_R f\frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}: z = g(x,y)$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\pm(-\frac{\partial g}{\partial x}\,\mathbf{i}-\frac{\partial g}{\partial y}\,\mathbf{j}+\,\mathbf{k})\,dx\,dy\,.$$

Flux of a vector field $\mathbf{F}$ through a surface $\mathscr{S}: G(x,y,z)=c$, parametrised by $(x,y)\in R$:

$$\iint_{\mathscr{S}} \mathbf{F}\bullet d\mathbf{S} = \iint_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_R \mathbf{F}\bullet\frac{\pm\nabla G}{\frac{\partial G}{\partial z}}\,dx\,dy\,.$$

Divergence theorem:

$$\iiint\limits_D \nabla\bullet\mathbf{F}\ dV = \oiint\limits_{\mathscr{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\ dS\,.$$

Stokes' theorem:

$$\iint\limits_{\mathscr{S}} (\nabla\times\mathbf{F})\bullet\hat{\mathbf{N}}\ dS = \oint\limits_{\mathscr{C}} \mathbf{F}\bullet d\mathbf{r}\,.$$

Formulas involving $\nabla = \frac{\partial}{\partial x}\,\mathbf{i}+\frac{\partial}{\partial y}\,\mathbf{j}+\frac{\partial}{\partial z}\,\mathbf{k}$:

$$\text{grad } f = \nabla f, \quad \text{div } \mathbf{F} = \nabla\bullet\mathbf{F}, \quad \text{curl } \mathbf{F} = \nabla\times\mathbf{F}\,.$$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.
Spherical coordinates: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.
Trigonometric formulas: $\sin 2\theta = 2\sin\theta\cos\theta$, $\quad\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

Exercise 1

$\mathcal{C}: \ \underset{\sim}{r}(t) = \cos 3t \ \underset{\sim}{i} + e^t \underset{\sim}{j} - \sin 3t \ \underset{\sim}{k}, \quad 0 \le t \le 2\pi$

(a) (i) $\mathcal{C}$ is closed if $\underset{\sim}{r}(0) = \underset{\sim}{r}(2\pi)$. We have

$$\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j}$$

$$\underset{\sim}{r}(2\pi) = \underset{\sim}{i} + e^{2\pi} \underset{\sim}{j}$$

Since $\underset{\sim}{r}(0) \neq \underset{\sim}{r}(2\pi)$, $\mathcal{C}$ is not closed.

(ii) $$\frac{d\underset{\sim}{r}}{dt} = -3 \sin 3t \ \underset{\sim}{i} + e^t \underset{\sim}{j} - 3 \cos 3t \ \underset{\sim}{k}$$

$t = \pi$: $$\frac{d\underset{\sim}{r}}{dt}(\pi) = e^{\pi} \underset{\sim}{j} + 3\underset{\sim}{k}$$

$$\left| \frac{d\underset{\sim}{r}}{dt}(\pi) \right| = \sqrt{(e^{\pi})^2 + 3^2} = \sqrt{e^{2\pi} + 9}$$

$$\Rightarrow \ \hat{\underset{\sim}{T}} = \frac{e^{\pi} \underset{\sim}{j} + 3\underset{\sim}{k}}{\sqrt{e^{2\pi} + 9}}.$$

(b) $\displaystyle\int_{C} x^2+y^2+z^2-1 \, ds$

$$= \int_0^{2\pi} \left[\cos^2(3t) + (e^t)^2 + \sin^2(3t) - 1\right] \left|\frac{dr}{dt}\right| dt$$

$$\left|\frac{dr}{dt}\right| = \sqrt{9\sin^2(3t) + (e^t)^2 + 9\cos^2(3t)}$$

$$= \sqrt{9 + e^{2t}}$$

$$= \int_0^{2\pi} e^{2t}\left(9 + e^{2t}\right)^{1/2} dt$$

$$= \frac{2}{3} \cdot \frac{1}{2} \cdot \left(9 + e^{2t}\right)^{3/2} \Big|_0^{2\pi} = \frac{1}{3}\left(9 + e^{4\pi}\right)^{3/2} - \frac{1}{3}(10)^{3/2}$$

(c) $\underset{\sim}{F}(x,y,z) = (\sin y + ze^x)\underset{\sim}{i} + (x\cos y + \cos z)\underset{\sim}{j}$
$+ (-y\sin z + e^x)\underset{\sim}{k}$

$$\frac{\partial \phi}{\partial x} = \sin y + ze^x \Rightarrow \phi(x,y,z) = x\sin y + ze^x + c_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = x\cos y + \cos z = x\cos y + \frac{\partial c_1}{\partial y}$$

$$\Rightarrow \frac{\partial c_1}{\partial y} = \cos z \Rightarrow c_1(y,z) = y\cos z + c_2(z)$$

$$\frac{\partial \phi}{\partial z} = -y\sin z + e^x = e^x - y\sin z + \frac{dc_2}{dz}$$

$\Rightarrow c_2(z) = 0$. (or a constant)

So $\phi(x,y,z) = x\sin y + y\cos z + ze^x$ is a scalar potential for $\underset{\sim}{F}$, and $\underset{\sim}{F}$ is conservative.

(d) $\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}(2\pi)) - \phi(\underset{\sim}{r}(0))$, since

$\underset{\sim}{F}$ has scalar potential ϕ.

$\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j} = (1, 1, 0)$

$\underset{\sim}{r}(2\pi) = \underset{\sim}{i} + e^{2\pi} \underset{\sim}{j} = (1, e^{2\pi}, 0)$.

$\phi(1,1,0) = \sin 1 + 1$

$\phi(1, e^{2\pi}, 0) = \sin(e^{2\pi}) + e^{2\pi}$

$\Rightarrow \int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \sin(e^{2\pi}) + e^{2\pi} - \sin(1) - 1$

Exercise 2

Transformation : $u = y, \quad v = x^2 + y.$

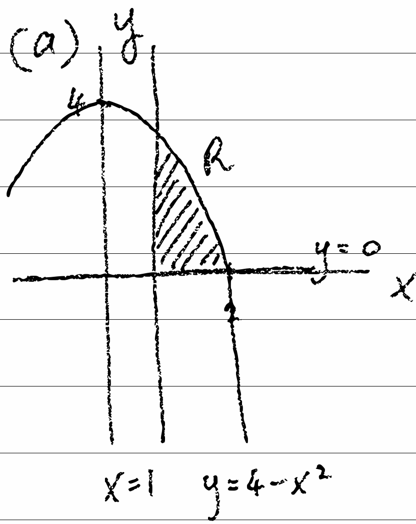
R : region in right half of xy-plane bounded by

① $y = 4 - x^2$

② $x = 1$

③ x-axis , $y = 0.$

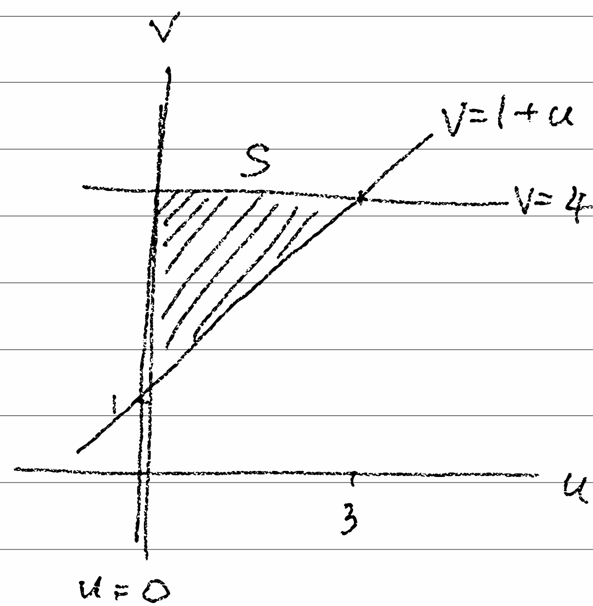
(a)



① $y = 4 - x^2 \Rightarrow x^2 + y = 4 \Rightarrow v = 4$

② $x = 1 \Rightarrow v = 1 + y = 1 + u$

③ $y = 0 \Rightarrow u = y = 0.$



(b)(i) $\iint_R xy(x^2+y)\,dx\,dy$

On the region R we have $x \geqslant 0$ and $y \geqslant 0$. Thus the integrand $xy(x^2+y) \geqslant 0$ everywhere on R. The integral of a function that is greater than or equal to zero is also greater than or equal to zero.

(ii) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 2x & 1 \end{vmatrix} = -2x$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2x}.$$

(iii) $\iint_R xy(x^2+y)\,dx\,dy = \iint_S xy(x^2+y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$= \iint_S xy(x^2+y)\left|-\frac{1}{2x}\right| du\,dv$

$= \iint_S xy(x^2+y)\,\frac{1}{2x}\,du\,dv$ (since $x>0$ on R)

$= \iint_S \frac{1}{2}\,y(x^2+y)\,du\,dv = \frac{1}{2}\iint_S uv\,du\,dv.$

Now S: $0 \le u \le 3$, $1+u \le v \le 4$.

$= \frac{1}{2}\int_0^3 \int_{1+u}^4 uv\,dv\,du = \frac{1}{2}\int_0^3 \frac{1}{2}uv^2\Big|_{1+u}^4 \,du$

$= \frac{1}{4}\int_0^3 16u - u(1+u)^2 \,du$

$$= \frac{1}{4}\int_0^3 16u - u - 2u^2 - u^3 \, du$$

$$= \frac{1}{4}\left[\frac{15}{2}u^2 - \frac{2}{3}u^3 - \frac{1}{4}u^4 \right]_0^3$$

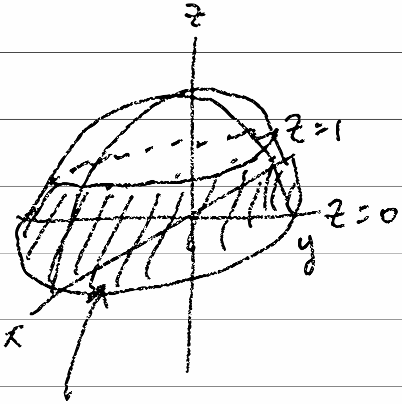
$$= \frac{1}{4}\left[\frac{15}{2}\cdot 9 - \frac{2}{3}\cdot 27 - \frac{1}{4}\cdot 81 \right]$$

$$= \frac{1}{4}\left[\frac{135}{2} - 18 - \frac{81}{4} \right]$$

$$= \frac{1}{4}\left[\frac{270 - 72 - 81}{4} \right] = \frac{117}{16}$$

Exercise 3

$S: \; x^2+y^2+z^2=4, \quad 0 \le z \le 1.$



R: part of S with $y \geq 0$.

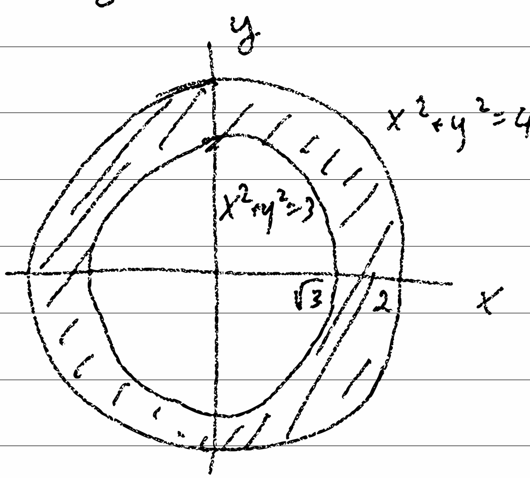
(a) (i) Let $f(x,-y,z) = f(x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$.

Since S is symmetric under reflection in the xz-plane, given by $(x,y,z) \mapsto (x,-y,z)$, and since f is <u>even</u> in the y-variable, the integral of f over the half of S with $y \le 0$ equals the integral of f over the half of S with $y \geq 0$. That is,

$$\iint_S f\,dS = \iint_{\substack{S \\ y\le 0}} f\,dS + \iint_{\substack{S \\ y\geq 0}} f\,dS = 2\iint_{\substack{S \\ y\geq 0}} f\,dS = 2\iint_R f\,dS.$$

(ii) $\iint_S x^2 z \, dS$. S meets xy-plane $z=0$ in circle $x^2+y^2=4$.
S meets plane $z=1$ in circle $x^2+y^2+1=4 \Rightarrow x^2+y^2=3$

Projection of S onto xy-plane is the region $D: 0 \le \theta \le 2\pi, \sqrt{3} \le r \le 2$.



$S: x^2+y^2+z^2=4$. Let $G(x,y,z)=x^2+y^2+z^2$, then $S: G(x,y,z)=4$ (level set).

$\underset{\sim}{\nabla} G = 2x\underset{\sim}{i} + 2y\underset{\sim}{j} + 2z\underset{\sim}{k}$,

$|\underset{\sim}{\nabla} G| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2}$
$= 2\sqrt{4} = 4$

$$\frac{\partial G}{\partial z} = 2z. \quad \text{So} \quad \frac{|\nabla G|}{\left|\frac{\partial G}{\partial z}\right|} = \frac{4}{2z} = \frac{2}{z}$$

$$\Rightarrow \iint_S x^2 z \, dS = \iint_D x^2 z \cdot 2/z \, dx\,dy = 2\iint_D x^2 \, dx\,dy$$

Use polar coords: $dx\,dy = r\,dr\,d\theta$

$x = r\cos\theta$

$$= 2\int_0^{2\pi}\int_{\sqrt{3}}^{2} r^2\cos^2\theta \, r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\cos^2\theta\,d\theta \cdot \left.\frac{r^4}{4}\right|_{\sqrt{3}}^{2}$$

$$= 2\int_0^{2\pi}\frac{1}{2}\left[\cos 2\theta + 1\right]d\theta \cdot 1/4\,(16-9)$$

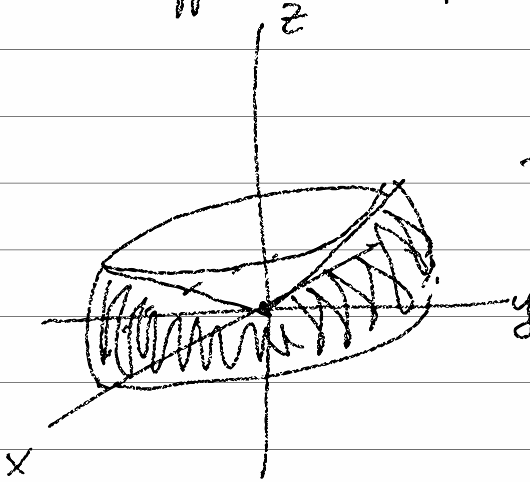
$$= 7/4\left(1/2\sin 2\theta + \theta\right]_0^{2\pi} = \frac{7\pi}{2}.$$

(b) T: region bounded by S, $z=0$, and cone $x^2+y^2=3z^2$.

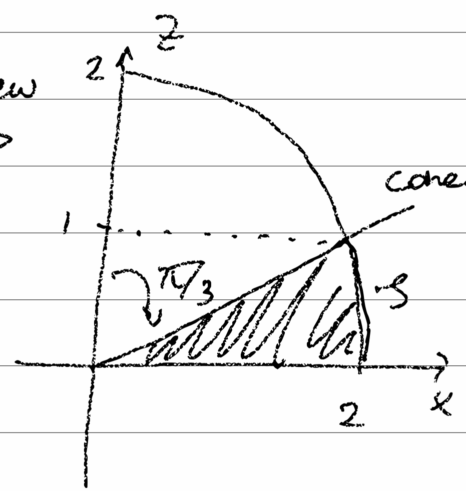
The cone meets S: $x^2+y^2+z^2=4$

$3z^2+z^2=4 \Rightarrow z^2=1$

$z=\pm 1$, but S is in upper half-space, so $z=1$.



side view $\longrightarrow$



Thus, in spherical coordinates:

$$T: 0 \le \theta \le 2\pi, \quad \pi/3 \le \phi \le \pi/2,$$

$$0 \le R \le 2$$

Then $\mathrm{vol}(T) = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 R^2 \sin\phi \, dR \, d\phi \, d\theta$

$= 2\pi \cdot (-\cos\phi)\Big|_{\pi/3}^{\pi/2} \cdot 8/3$

$= \frac{16\pi}{3} \cdot \left(0 + \cos \pi/3\right) = \frac{8\pi}{3}.$

Exercise 4

$$\underset{\sim}{F}(x,y,z) = (y - xz^2)\underset{\sim}{i} + (\sin y + \tfrac{1}{2}x^2 z)\underset{\sim}{j} + (\tfrac{1}{2}x^2 y + e^x)\underset{\sim}{k}$$

(a) $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = -z^2 + \cos y + 0$

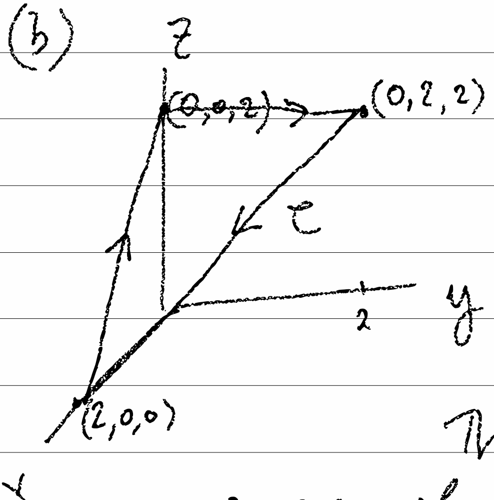
$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ y - xz^2 & \sin y + \tfrac{1}{2}x^2 z & \tfrac{1}{2}x^2 y + e^x \end{vmatrix}$$

$$= \underset{\sim}{i}\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x^2\right) + \underset{\sim}{j}\left(-2xz - xy - e^x\right)$$

$$+ \underset{\sim}{k}\left(xz - 1\right)$$

$$= \underset{\sim}{j}\left(-2xz - xy - e^x\right) + \underset{\sim}{k}\left(xz - 1\right).$$

(b)



Let T be the triangular surface with vertices $(2,0,0)$, $(0,0,2)$, $(0,2,2)$, with the downward pointing unit normal $\hat{\underset{\sim}{N}}$. Then, $\mathcal{C}$ is the boundary curve of T.

By Stokes' theorem,

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_{T} (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} \, dS .$$

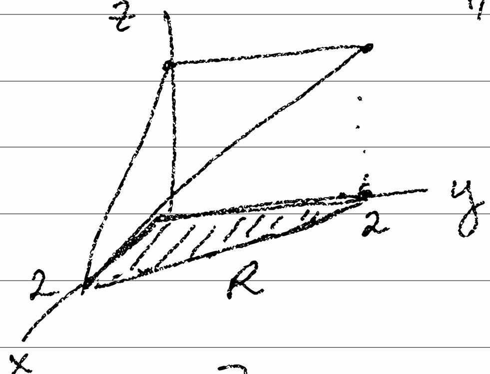
Now T lies in the plane $x + z = 2$. Let $G(x,y,z) = x + z$; then T lies in surface $G(x,y,z) = 2$.

$$\underset{\sim}{\nabla} G = (1,0,1), \qquad \frac{\partial G}{\partial z} = 1.$$

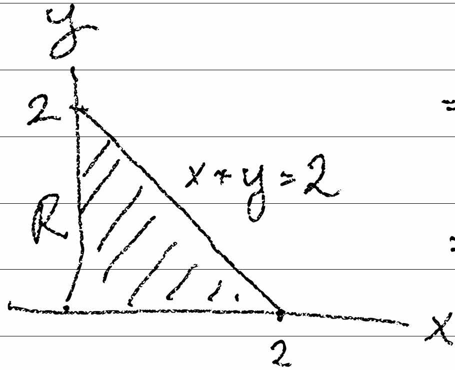
to get downward normal.

$$\hat{\underset{\sim}{N}} \, dS = \pm \frac{\nabla G}{\partial G / \partial z} \, dx\,dy = -(1,0,1)\, dx\,dy$$
$$= (-1,0,-1)\, dx\,dy.$$

Projection of T onto xy-plane is the triangle



$R: \; 0 \le x \le 2$
$\quad\;\; 0 \le y \le 2-x.$



And $(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS$

$= (1 - xz)\, dx\, dy$ $\begin{pmatrix} z+x=2 \\ \text{on } T \end{pmatrix}$

$= \big(1 - x(2-x)\big)\, dx\, dy$

$= (1 - 2x + x^2)\, dx\, dy$

So $\displaystyle\iint_T (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS$

$\displaystyle = \int_0^2 \int_0^{2-x} 1 - 2x + x^2 \; dy\, dx$

$$= \int_0^2 y - 2xy + x^2 y \Big|_{y=0}^{2-x} dx$$

$$= \int_0^2 2 - x - 2x(2-x) + x^2(2-x)\, dx$$

$$= \int_0^2 2 - x - 4x + 2x^2 + 2x^2 - x^3\, dx$$

$$= \int_0^2 2 - 5x + 4x^2 - x^3\, dx$$

$$= 2x - \tfrac{5}{2}x^2 + \tfrac{4}{3}x^3 - \tfrac{1}{4}x^4 \Big|_0^2$$

$$= 4 - 10 + \frac{32}{3} - 4 = \frac{32}{3} - \frac{30}{3} = \tfrac{2}{3}.$$