ASSIGNMENT 1 – MAT300 VEKTORANALYSE

This assignment is not compulsory, but if you submit by Friday 26. October 2018 then I will check your answers and provide feedback.

(1) Let $R \subset \mathbb{R}^3$ be the region above the xy-plane bounded by the two surfaces $z = 6 - x^2 - y^2$ and $z^2 = x^2 + y^2$. Find the volume of R.

(2) Compute the integral

$$\int_0^1 dx \int_x^1 x e^{y^3}\, dy\,.$$

(3) Let $D \subset \mathbb{R}^2$ be the region bounded by the four curves $y = x^3 - 4$, $y = x^3 + 4$, $x + y = 6$, and $x + y = 0$. Use an appropriate change of variables to compute the integral

$$\iint_D (3x^3 + 3x^2 y + x + y)\, dA\,.$$

(4) Let $\mathcal{C} \subset \mathbb{R}^3$ be the piece of the curve of intersection of the cylinder $xy = 1$ and the plane $2x - 3y - z = -1$ from the point $(\frac{1}{2}, 2, -4)$ to the point $(3, \frac{1}{3}, 6)$.
   (a) Find a parametrisation for $\mathcal{C}$.
   (b) Suppose that the pointwise density (mass per unit length) of a piece of wire in the shape of $\mathcal{C}$ is given by the formula $\rho(x, y, z) = 5x^5 + \frac{6}{y^3}$. Compute the mass of the wire.

(5) Let $f(x, y, z) = x^2 - y^2 - z$.
   (a) Find $\nabla f(x, y, z)$ (the gradient of f).
   (b) Find the directional derivative of f at the point $(-1, 3, 2)$ in the direction $\mathbf{i} - 2\mathbf{k}$.
   (c) Find the maximal rate of increase of f at the point $(0, 0, 0)$ and give the direction in which it occurs.
   (d) Find the equation of the level surface of f through the point $(1, -2, 0)$ and give the equation of the tangent plane to the level surface at that point.

(6) Consider the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + y\mathbf{j}$.
   (a) Find and describe the field lines of $\mathbf{F}$.
   (b) Check that $\mathbf{F}$ satisfies the necessary condition for being conservative (as discussed in lectures).
   (c) Show that $\mathbf{F}$ is indeed conservative by finding a potential $\phi(x, y)$ for $\mathbf{F}$.
   (d) Describe the equipotential curves of $\mathbf{F}$. Sketch a few field lines and equipotential curves and convince yourself that they always intersect perpendicularly.
   (e) Compute the line integral of $\mathbf{F}$ from $P_0 = (0, 2)$ to $P_1 = (2, 0)$ along the following curves:
      (i) The straight line segment from P_0 to P_1.
      (ii) The curve consisting of the two straight line segments from P_0 to the origin and then the origin to P_1.
      (iii) The quarter-arc of the circle of radius 2, centred at the origin, from P_0 to P_1 (in the clockwise direction).
   (f) Compute the line integral of $\mathbf{F}$ from P_0 to P_1 using your potential function ϕ, and check it agrees with your answers to part (e).

Date: October 8, 2018.