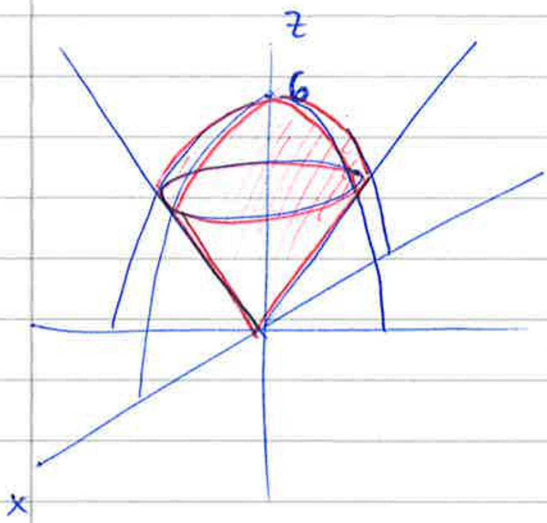
Assignment 1 Solutions

(1) $R \subset \mathbb{R}^3$ given by $z \geq 0$, between $z = 6 - x^2 - y^2$ and $z^2 = x^2 + y^2$.



Use cylindrical coords, (r, θ, z).
Then $z = 6 - r^2$ and $z^2 = r^2$
$\Rightarrow z = r$
(since $z \geq 0$).

Intersection: $6 - r^2 = r$
$\Rightarrow r^2 + r - 6 = 0$
$\Rightarrow (r+3)(r-2) = 0$
$\Rightarrow r = 2 \quad (r \geq 0)$

Limits:
$0 \leq \theta \leq 2\pi$
$0 \leq r \leq 2$
$r \leq z \leq 6 - r^2$

$dV = r\,dr\,d\theta\,dz$

$$\text{Vol}(R) = \int_0^{2\pi} d\theta \int_0^2 dr \int_r^{6-r^2} r\,dz$$

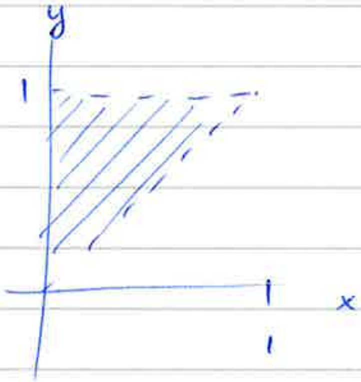
$$= 2\pi \int_0^2 rz \Big|_{z=r}^{6-r^2} dr = 2\pi \int_0^2 6r - r^3 - r^2\,dr$$

$$= 2\pi \left[3r^2 - \tfrac{1}{4}r^4 - \tfrac{1}{3}r^3 \right]_0^2$$

$$= 2\pi \left[12 - 4 - \tfrac{8}{3} \right] = \frac{32\pi}{3}.$$

(2) $\int_0^1 dx \int_x^1 x e^{y^3} dy = I$

Cannot integrate like this, try changing order of integration. Currently, $0 \le x \le 1$, $x \le y \le 1$:



$0 \le y \le 1$, $0 \le x \le y$.

$$I = \int_0^1 dy \int_0^y x e^{y^3} d$$

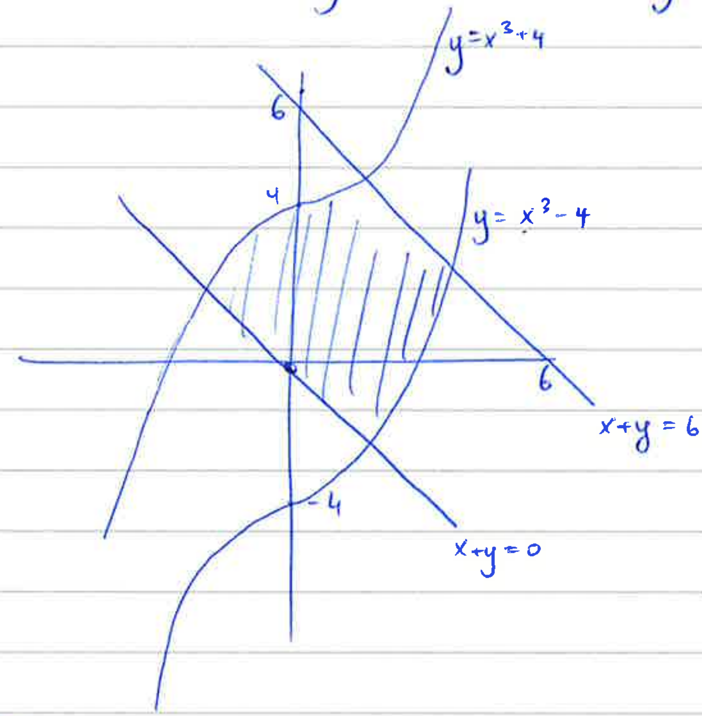
$$= \int_0^1 \frac{1}{2} x^2 e^{y^3} \Big|_{x=0}^{y} dy$$

$$= \int_0^1 \frac{1}{2} y^2 e^{y^3} dy$$

Now $\frac{d}{dy} e^{y^3} = 3y^2 e^{y^3} \Rightarrow \int y^2 e^{y^3} dy = \frac{1}{3} e^{y^3}$

$$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{3} e^{y^3} \Big|_0^1 = \frac{1}{6}(e-1)$$

(3) $D \subset \mathbb{R}^2$: $y = x^3 - 4$, $y = x^3 + 4$, $x+y = 6$, $x+y = 0$.



Try $u = x^3 - y$, $v = x+y$. Then limits change to
R: $-4 \le u \le 4$, $0 \le v \le 6$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x^2 & -1 \\ 1 & 1 \end{vmatrix} = 3x^2 + 1$$

$\Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{3x^2+1}$, not clear how to write in terms of u,v at this point, so we continue & see what happens.

$\Rightarrow \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\dfrac{1}{3x^2+1}\right| = \dfrac{1}{3x^2+1}$.

$$I = \iint_D (3x^3 + 3x^2 y + x + y)\, dA = \iint_R (3x^3 + 3x^2 y + x + y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

Now $3x^3 + 3x^2 y + x + y = 3x^2(x+y) + 1(x+y) = (3x^2+1)(x+y)$

So $(3x^3 + 3x^2y + x + y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

$$= (3x^2+1)(x+y)\cdot\frac{1}{3x^2+1} = x+y = v.$$

Thus $I = \iint_R v \, du\, dv$

$$= \int_{-4}^{4} du \int_0^6 v \, dv$$

$$= 8 \cdot \tfrac{1}{2} v^2 \Big|_0^6 = 8 \cdot \frac{36}{2} = 144.$$

(4) $\mathcal{C}$: $xy=1$ intersect $2x-3y-z=-1$, from $(\frac{1}{2}, 2, -4)$ to $(3, \frac{1}{3}, 6)$.

let $x=t$, $\Rightarrow y = 1/t \Rightarrow z = 2t - \frac{3}{t} + 1$, $t \in [\frac{1}{2}, 3]$.

(a) $\underset{\sim}{r}(t) = t\,\underset{\sim}{i} + \frac{1}{t}\,\underset{\sim}{j} + (2t - \frac{3}{t} + 1)\,\underset{\sim}{k}$, $t \in [\frac{1}{2}, 3]$.

$\underset{\sim}{v}(t) = \underset{\sim}{i} - \frac{1}{t^2}\,\underset{\sim}{j} + (2 + \frac{3}{t^2})\,\underset{\sim}{k}$

(b) $\rho(x, y, z) = 5x^5 + \frac{6}{y^3}$

$\Rightarrow \rho(\underset{\sim}{r}(t)) = 5t^5 + 6t^3$

$$\text{mass} = \int_{\mathcal{C}} \rho(\underset{\sim}{r}(t))\, v\, dt = \int_{\frac{1}{2}}^{3} (5t^5 + 6t^3)\frac{1}{t^2}(5t^4 + 12t^2 + 10)^{1/2}\, dt$$

$$\left[\begin{aligned} v &= \left(1 + \frac{1}{t^4} + (2 + \frac{3}{t^2})^2\right)^{1/2} = \left(1 + \frac{1}{t^4} + 4 + \frac{12}{t^2} + \frac{9}{t^4}\right)^{1/2} \\ &= \frac{1}{t^2}(5t^4 + 12t^2 + 10)^{1/2} \end{aligned} \right]$$

$$\text{mass} = \int_{1/2}^{3} (5t^3 + 6t)(5t^4 + 12t^2 + 10)^{1/2}\, dt$$

$$= \frac{2}{3} \cdot \frac{1}{4} (5t^4 + 12t^2 + 10)^{3/2} \Big|_{1/2}^{3}$$

$$= \frac{1}{6}\left[(405 + 108 + 10)^{3/2} - (\frac{5}{16} + 3 + 10)^{3/2}\right]$$

$$= \frac{1}{6}\left[(523)^{3/2} - (\frac{213}{16})^{3/2}\right] \approx 1985.34$$

(5) $f(x,y,z) = x^2 - y^2 - z$.

(a) $\underset{\sim}{\nabla} f = 2x\underset{\sim}{i} - 2y\underset{\sim}{j} - \underset{\sim}{k}$

(b) $\underset{\sim}{v} = \underset{\sim}{i} - 2\underset{\sim}{k} \Rightarrow |\underset{\sim}{v}| = \sqrt{1+4} = \sqrt{5} \Rightarrow \hat{\underset{\sim}{v}} = \frac{1}{\sqrt{5}}\underset{\sim}{i} - \frac{2}{\sqrt{5}}\underset{\sim}{k}$

$\underset{\sim}{\nabla} f(-1,3,2) = -2\underset{\sim}{i} - 6\underset{\sim}{j} - \underset{\sim}{k}$

$D_{\hat{\underset{\sim}{v}}} f(-1,3,2) = \underset{\sim}{\nabla} f(-1,3,2) \cdot \hat{\underset{\sim}{v}} = \frac{-2}{\sqrt{5}} + \frac{2}{\sqrt{5}} = 0.$

(c) $\underset{\sim}{\nabla} f(0,0,0) = -\underset{\sim}{k}$, and $\underset{\sim}{\nabla} f$ points in direction of maximal rate of increase of f.
So $\hat{\underset{\sim}{v}} = -\underset{\sim}{k}$ is the direction.
Then $D_{\hat{\underset{\sim}{v}}} f(0,0,0) = \underset{\sim}{\nabla} f \cdot \hat{\underset{\sim}{v}} = 1.$

(d) $f(1,-2,0) = 1 - 4 - 0 = -3 \Rightarrow f(x,y,z) = -3 = x^2 - y^2 - z$ is level surface through $(1,-2,0)$.

$\underset{\sim}{\nabla} f(1,-2,0) = 2\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ normal vector to level surface at $(1,-2,0)$, hence also normal vector to tangent plane at that point.

Tangent plane has equation $2x + 4y - z = 2(1) + 4(-2) - 0$
$= -6.$

(6) $\underset{\sim}{F}(x,y) = \frac{x}{2}\underset{\sim}{i} + y\underset{\sim}{j}$

(a) $\frac{dx}{x/2} = \frac{dy}{y} \Rightarrow 2\frac{dx}{x} = \frac{dy}{y} \Rightarrow 2\ln x + \ln c = \ln y$

$\Rightarrow \ln x^2 + \ln c = \ln y \Rightarrow \ln(cx^2) = \ln y$

$\Rightarrow y = cx^2$ all parabolas through $(0,0)$
(straight line $y=0$ if $c=0$).

(b) $\frac{\partial F_2}{\partial x} = 0 = \frac{\partial F_1}{\partial y}$ ✓

(c) $\frac{\partial \phi}{\partial x} = \frac{x}{2} \qquad \frac{\partial \phi}{\partial y} = y$

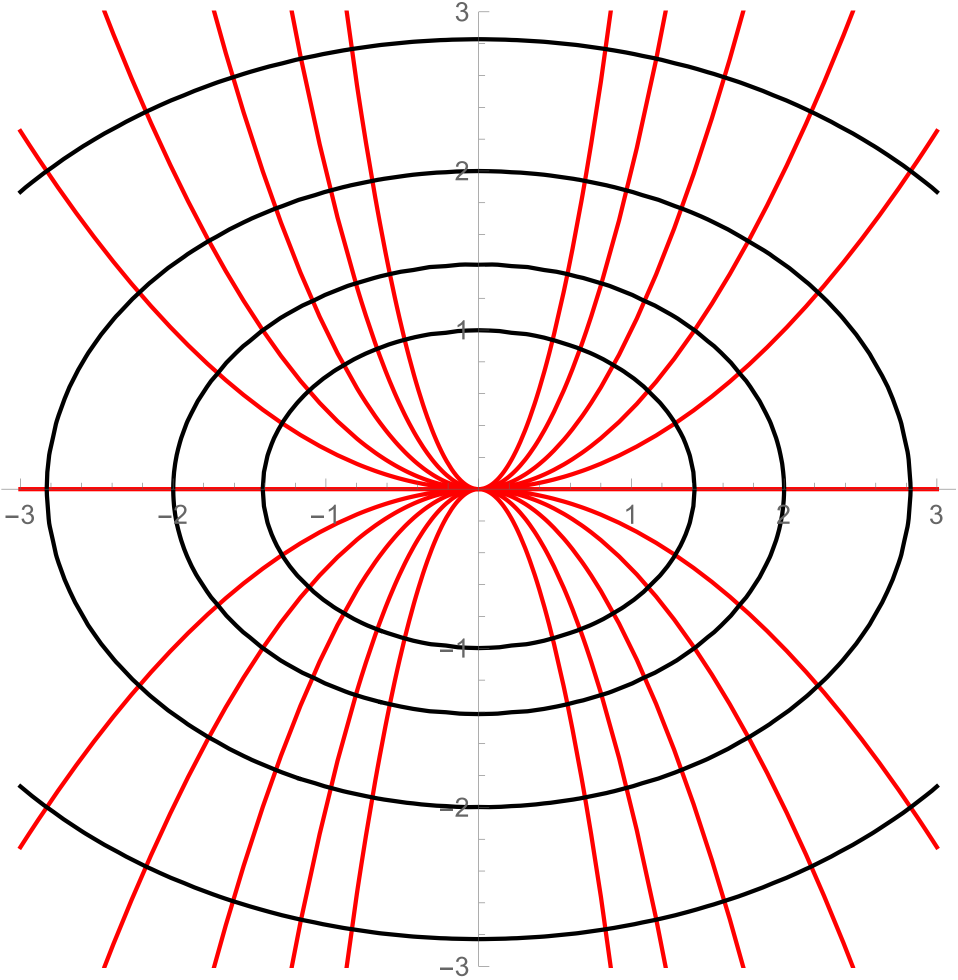
$\Rightarrow \phi = \frac{x^2}{4} + c(y) \Rightarrow \frac{\partial \phi}{\partial y} = \frac{\partial c}{\partial y} = y \Rightarrow c(y) = \frac{1}{2}y^2 + d.$

$\phi = \frac{x^2}{4} + \frac{1}{2}y^2 + d.$

(d) equipotential curves are level sets of ϕ.

$\phi(x,y) = k = \frac{x^2}{4} + \frac{y^2}{2} = \begin{cases} \text{empty set} & \text{if } k<0 \\ (0,0) & \text{if } k=0 \\ \text{ellipse} & \text{if } k>0. \end{cases}$

$1 = \left(\frac{x}{\sqrt{k}\,2}\right)^2 + \left(\frac{y}{\sqrt{k2}}\right)^2$


3
2
1
−1
−2
−3
−3
−2
−1
1
2
3

(e) $P_0 = (0,2)$ to $P_1 = (2,0)$ along:

(i) line segment:

$$\underset{\sim}{r}(t) = (0,2) + t((2,0)-(0,2))$$
$$= (0,2) + t(2,-2) \qquad t \in [0,1]$$
$$= 2t\underset{\sim}{i} + (2-2t)\underset{\sim}{j} \qquad t \in [0,1].$$

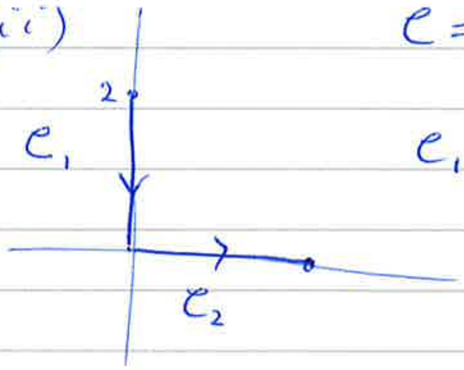
$$d\underset{\sim}{r} = 2dt\,\underset{\sim}{i} - 2dt\,\underset{\sim}{j}$$

$$\underset{\sim}{F}(\underset{\sim}{r}(t)) = t\underset{\sim}{i} + (2-2t)\underset{\sim}{j}$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = (2t - 4 + 4t)dt = (6t-4)dt$$

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 6t - 4 \; dt = 3 - 4 = -1.$$

(ii)



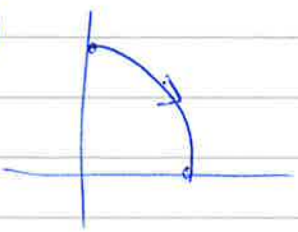
$$C = C_1 + C_2$$

$$C_1: \; d\underset{\sim}{r} = dy\underset{\sim}{j}, \quad \int_2^0 y\,dy = -2$$

$$C_2: \; d\underset{\sim}{r} = dx\underset{\sim}{i}, \quad \int_0^2 \frac{x}{2}\,dx = 1$$

$$\Rightarrow \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{C_1} + \int_{C_2} = -2 + 1 = -1$$

(iii)



$x = 2\cos\theta \quad y = 2\sin\theta, \quad \theta = \pi/2$ to 0.

$$d\underset{\sim}{r} = -2\sin\theta\, d\theta\, \underset{\sim}{i} + 2\cos\theta\, d\theta\, \underset{\sim}{j}$$

$$\underset{\sim}{F}(\underset{\sim}{r}(\theta)) = \cos\theta\, \underset{\sim}{i} + 2\sin\theta\, \underset{\sim}{j}$$

$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = (-2\sin\theta\cos\theta + 4\sin\theta\cos\theta)\,d\theta = 2\sin\theta\cos\theta\,d\theta$$

$\sin 2\theta = 2 \sin\theta \cos\theta$

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{\pi/2}^{0} \sin 2\theta \, d\theta = -\frac{1}{2} \cos(2\theta) \Big|_{\pi/2}^{0}$$

$$= -\frac{1}{2}\left(1 - (-1)\right) = -1 .$$

(f) $$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \phi(P_1) - \phi(P_0)$$

$$= \phi(2,0) - \phi(0,2)$$

$$= \frac{2^2}{4} + 0 - \left(0 + \frac{2^2}{2}\right)$$

$$= 1 - 2 = -1 \quad \checkmark$$