ASSIGNMENT 2 – MAT300 VEKTORANALYSE

This assignment is NOT COMPULSORY, but if you submit by Friday 23. November 2018 then I will check your answers and provide feedback.

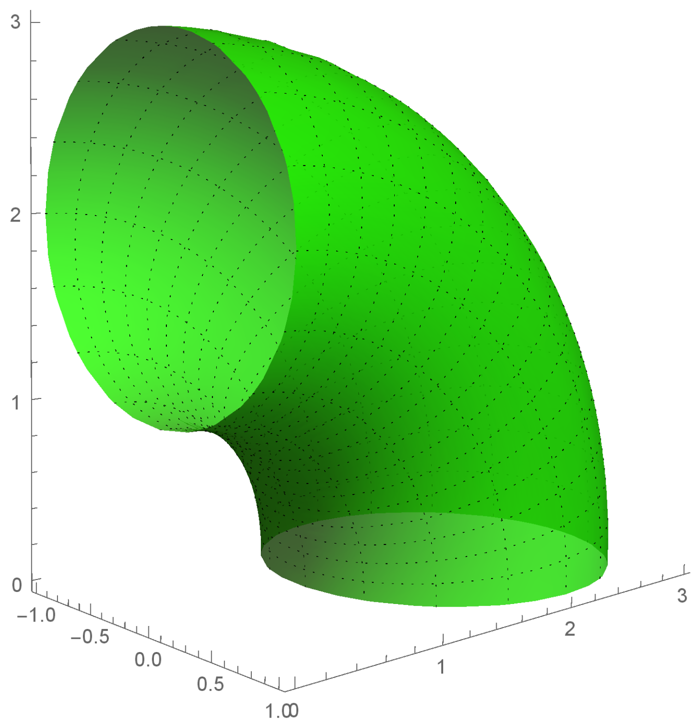
(1) Let $R \subset \mathbb{R}^2$ be the triangle with vertices at the points $(-1, 2)$, $(-2, -2)$, and $(3, 0)$, with the disc of radius 1 centred at the origin removed.
  (a) Use Green's theorem to compute the area of R.
  (b) Check your answer by computing the area of R using another method.

(2) Let $D \subset \mathbb{R}^2$ be the rainbow-shaped region defined by $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$. Let $\mathscr{C}$ be the boundary of D, equipped with the positive orientation. Let

$$I = \oint_{\mathscr{C}} \sin\sqrt{x^2 + y^2}\,(dx + dy)\,.$$

  (a) Compute I explicitly as a path integral.
  (b) Use Green's theorem to compute I and check your answer agrees with (a).

(3) Let $\mathscr{S} \subset \mathbb{R}^3$ be the surface shown below.



The surface $\mathscr{S}$ is one-quarter of the torus (doughnut) obtained by taking a circle of radius 1 in the xy-plane and rotating it along a circular path of radius 2 in the yz-plane. (In the picture, the z-axis is vertical, the y-axis goes from 0 to 3, and the x-axis goes from -1 to 1.)
  (a) Show that $\mathscr{S}$ can be parametrised as

$$\mathbf{r}(u, v) = \cos(v)\mathbf{i} + \cos(u)(2 + \sin(v))\mathbf{j} + \sin(u)(2 + \sin(v))\mathbf{k}\,,$$

  where $0 \leq u \leq \pi/2$, $0 \leq v \leq 2\pi$.

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(b) Using this parametrisation, compute the area of $\mathscr{S}$.

(c) Define the scalar function

$$\rho(x, y, z) = x + x\sqrt{y^2 + z^2}\,.$$

Compute $\iint_{\mathscr{S}} \rho\, dS$. Why did you expect this result?

(d) Orient $\mathscr{S}$ with the outward-pointing unit normal vector field $\hat{\mathbf{N}}$. Find an expression for $d\mathbf{S} = \hat{\mathbf{N}} dS$.

(e) Define a vector field $\mathbf{G}(x, y, z)$ by

$$\begin{aligned}\mathbf{G}(x, y, z) = {} & [-e^x(e^y + e^{-z}) + x(\sin y + e^{-z})]\mathbf{i} \\ & + (ye^{x-z} + \cos y)\mathbf{j} + (ze^{x+y} + e^{-z})\mathbf{k}\,.\end{aligned}$$

Try to write down the integral for the flux of $\mathbf{G}$ through $\mathscr{S}$, when $\mathscr{S}$ is oriented with the outward-pointing unit normal vector field. (Go as far as you can, but don't try to compute the integral. Just convince yourself that it's very complicated and messy.)

(f) Show that $\nabla \bullet \mathbf{G} = 0$.

(g) Use the divergence theorem to compute the flux of $\mathbf{G}$ through $\mathscr{S}$.

(h) Define a vector field $\mathbf{F}(x, y, z)$ by

$$\mathbf{F}(x, y, z) = (-ye^{x-z} - ze^{x+y})\mathbf{i} + e^{-z}(x - e^x)\mathbf{j} - (e^{x+y} + x\cos y)\mathbf{k}\,.$$

Show that $\nabla \times \mathbf{F} = \mathbf{G}$.

(i) Use Stokes' theorem to convert the flux integral for $\mathbf{G}$ through $\mathscr{S}$ into a path integral for $\mathbf{F}$. Try your best to compute this integral, and check your result agrees with your answer to (g).

Hint: If you get stuck performing the integrals, try differentiating

$$e^{\cos t}(-2 - \sin t) \quad \text{and} \quad e^{\cos t}(-2 + \sin t)\,.$$

Feel free to also use Wolfram Alpha, Mathematica, or Maple.