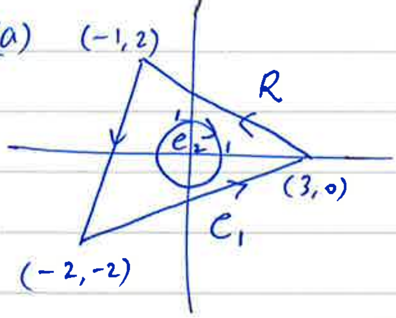
Assignment 2 Solutions.

(1) (a)



(a) $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ with orientation as shown.

$\mathcal{C}_1 = \mathcal{C}_{11} + \mathcal{C}_{12} + \mathcal{C}_{13}$ edges of triangle.

$\mathcal{C}_{11}$: $\underset{\sim}{r}(t) = (-2,-2) + t((3,0)-(-2,-2))$
$= (-2+5t)\underset{\sim}{i} + (-2+2t)\underset{\sim}{j}, \quad t \in [0,1].$

$\mathcal{C}_{12}$: $\underset{\sim}{r}(t) = (3-4t)\underset{\sim}{i} + 2t\underset{\sim}{j}, \quad t \in [0,1].$

$\mathcal{C}_{13}$: $\underset{\sim}{r}(t) = (-1-t)\underset{\sim}{i} + (2-4t)\underset{\sim}{j}, \quad t \in [0,1].$

$\mathcal{C}_2$: $\underset{\sim}{r}(t) = \cos t\,\underset{\sim}{i} - \sin t\,\underset{\sim}{j}, \quad t \in [0, 2\pi].$

Let $\underset{\sim}{F}(x,y) = \frac{1}{2}(-y\underset{\sim}{i} + x\underset{\sim}{j})$, then Green's theorem $\Rightarrow$

$$\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R 1 \cdot dx\,dy = \text{Area}(R).$$

Now $\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \left(\int_{\mathcal{C}_{11}} + \int_{\mathcal{C}_{12}} + \int_{\mathcal{C}_{13}} + \oint_{\mathcal{C}_2}\right) \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$$\int_{\mathcal{C}_{11}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_{\mathcal{C}_{11}} F_1\,dx + F_2\,dy = \int_0^1 \left(F_1(\underset{\sim}{r}(t))\frac{dx}{dt} + F_2(\underset{\sim}{r}(t))\frac{dy}{dt}\right) dt$$

$$= \frac{1}{2}\int_0^1 -(-2+2t)\cdot 5 + (-2+5t)\cdot 2 \quad dt$$

$$= \frac{1}{2}\int_0^1 6\,dt = 3$$

$$\int_{\mathcal{C}_{12}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \frac{1}{2}\int_0^1 -2t\cdot(-4) + (3-4t)\cdot 2 \quad dt = \frac{1}{2}\int_0^1 6\,dt = 3$$

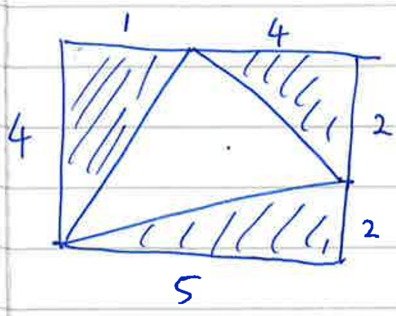
$$\int_{\mathcal{C}_{13}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \frac{1}{2}\int_0^1 -(2-4t)\cdot(-1) + (-1-t)\cdot(-4)\,dt = \frac{1}{2}\int_0^1 6\,dt = 3$$

$$\int_{C_2} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \frac{1}{2} \int_0^{2\pi} \sin t \cdot (-\sin t) + \cos t(-\cos t) \, dt$$

$$= \frac{1}{2} \int_0^{2\pi} (-1) \, dt = -\pi .$$

Thus $\text{area}(R) = 3 + 3 + 3 - \pi = 9 - \pi$.

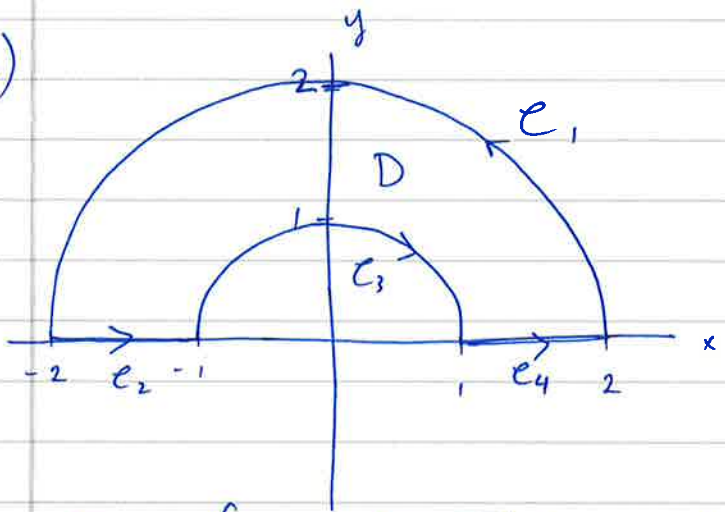
(b) To find area of the triangle, easiest to draw the box containing it and subtract the three smaller triangles:



$$20 - \frac{1}{2}[1 \cdot 4 + 4 \cdot 2 + 2 \cdot 5]$$
$$= 20 - \frac{1}{2}[22] = 9.$$

So area of $R = 9 - \pi \cdot 1^2 = 9 - \pi$.

(2)



$$I = \oint_{e} \sin\sqrt{x^2+y^2}\,(dx+dy) = \oint_{e} \sin\sqrt{x^2+y^2}\,dx + \sin\sqrt{x^2+y^2}\,dy.$$

(a) Write $e = e_1 + e_2 + e_3 + e_4$ where

$e_1 : \underset{\sim}{r}(t) = 2\cos\theta\,\underset{\sim}{i} + 2\sin\theta\,\underset{\sim}{j}, \quad 0 \le \theta \le \pi$

$e_2 : \underset{\sim}{r}(t) = (-2+t)\,\underset{\sim}{i}, \quad t \in [0,1]$

$e_3 : \underset{\sim}{r}(t) = -\cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j}. \quad 0 \le \theta \le \pi$

$e_4 : \underset{\sim}{r}(t) = (1+t)\,\underset{\sim}{i}, \quad t \in [0,1].$

$$I = \oint_e = \int_{e_1} + \int_{e_2} + \int_{e_3} + \int_{C_4}.$$

On e_2 : $x = (-2+t), \quad y = 0$

$dx = dt, \quad dy = 0$

$\sin\sqrt{x^2+y^2} = \sin\sqrt{x^2} = \sin|x| = \sin(2-t)$ (since $-2+t<0$, $t \in [0,1]$)

Get : $\int_{e_2} \sin(2-t)\,dt = \int_0^1 \sin(2-t)\,dt = \cos(2-t)\Big|_0^1$

$= \cos(1) - \cos(2)$

On e_4 : $x = 1+t \quad y = 0$

$dx = dt \quad dy = 0$

$\sin\sqrt{x^2+y^2} = \sin(1+t),$

get $\int_{e_4} \sin(1+t)\,dt = -\cos(1+t)\Big|_0^1 = -\cos(2) + \cos(1)$

On e_1: $x = 2\cos\theta$ $\quad y = 2\sin\theta$

$dx = -2\sin\theta \, d\theta \quad dy = 2\cos\theta \, d\theta$

$\sin\sqrt{x^2+y^2} = \sin 2$

get $\displaystyle\int_{e_1} \sin 2(-2\sin\theta + 2\cos\theta)\, d\theta = 2\sin 2 \int_0^{\pi} \cos\theta - \sin\theta \, d\theta$

$= 2\sin 2 \left[\sin\theta + \cos\theta \right]_0^{\pi} = 2\sin 2\left[0 + (-1) - (0+1)\right]$

$= -4\sin 2.$

On e_3: $x = -\cos\theta$ $\quad y = \sin\theta$

$dx = \sin\theta \, d\theta \quad dy = \cos\theta \, d\theta$

$\sin\sqrt{x^2+y^2} = \sin 1$

get $\displaystyle\int_{e_1} \sin(1)\cdot(\sin\theta + \cos\theta)\, d\theta = \sin(1)\int_0^{\pi} \sin\theta + \cos\theta \, d\theta$

$= \sin(1)\cdot(-\cos\theta + \sin\theta)_0^{\pi} = 2\sin 1$

So $\quad I = -4\sin 2 + \cos(1) - \cos(2) + 2\sin 1 + \cos(1) - \cos(2)$

$= 2\sin(1) - 4\sin(2) + 2\cos(1) - 2\cos(2).$

(b) By Green's theorem,

$$I = \iint_D \left(\frac{\partial \sin\sqrt{x^2+y^2}}{\partial x} - \frac{\partial \sin\sqrt{x^2+y^2}}{\partial y} \right) dx\, dy$$

$$\frac{\partial}{\partial x}\sin\sqrt{x^2+y^2} = \cos\left(\sqrt{x^2+y^2}\right)\cdot \frac{1}{2}\cdot \frac{2x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial}{\partial y} \sin\sqrt{x^2+y^2} = \cos\left(\sqrt{x^2+y^2}\right)\cdot \tfrac{1}{2}\cdot \frac{2y}{\sqrt{x^2+y^2}}.$$

$$I = \iint_D \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,(x-y)\,dx\,dy.$$

Use polar coordinates: $D: 0 \le \theta \le \pi, \quad 1 \le r \le 2$

$$I = \int_0^{\pi}\int_1^2 \frac{\cos(r)}{r}\cdot(r\cos\theta - r\sin\theta)\cdot \boxed{\overset{dx\,dy=}{r\,dr\,d\theta}}$$

$$= \int_0^{\pi}\int_1^2 (\cos\theta - \sin\theta)\, r\cos r\,dr\,d\theta$$

$$= \int_0^{\pi}(\cos\theta - \sin\theta)\,d\theta \cdot \int_1^2 r\cos r\,dr$$

Use integration by parts to find $\int r\cos r\,dr$:

let $u = r$, $\quad dv = \cos r\,dr$

$\Rightarrow \quad du = dr \quad \Rightarrow \quad v = \sin r$

$$\int r\cos r\,dr = \int u\,dv = uv - \int v\,du = r\sin r - \int \sin r\,dr$$

$$= r\sin r + \cos r.$$

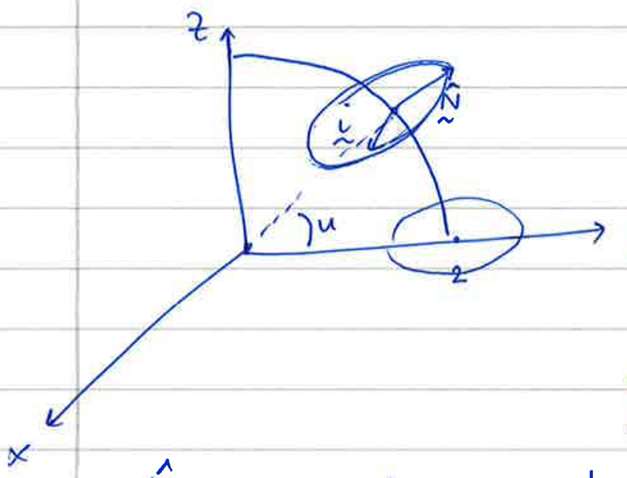
$$\Rightarrow I = (\sin\theta + \cos\theta)\Big|_0^{\pi}\cdot(r\sin r + \cos r)\Big|_1^2$$

$$= -2\cdot(2\sin 2 + \cos 2 - \sin 1 - \cos 1).$$

$$= 2\sin 1 - 4\sin 2 + 2\cos 1 - 2\cos 2$$ ✓

(3) (a) First of all, the quarter-circle of radius 2 in the yz-plane has parametrisation
$$\mathbf{r}_c(u) = 2\cos u\,\mathbf{j} + 2\sin u\,\mathbf{k}, \qquad 0 \le u \le \pi/2.$$
(u is angle from +ve y-axis.)

At each point $\mathbf{r}_c(u)$, we need to draw a circle of radius 1, with the correct orientation.



This circle can be produced by taking linear combinations of the vectors $\mathbf{i}$ and $\hat{\mathbf{N}}$, where $\hat{\mathbf{N}}$ is the normal vector to the quater circle in the yz-plane. We have $\hat{\mathbf{N}} = \cos u\,\mathbf{j} + \sin u\,\mathbf{k}$. Then the required circle is given by $\mathbf{r} = \mathbf{r}_c(u) + \cos v\,\hat{\mathbf{i}} + \sin v\,\hat{\mathbf{N}}$, $v \in [0, 2\pi]$. That is, the surface has param:

$$\mathbf{r}(u,v) = 2\cos u\,\mathbf{j} + 2\sin u\,\mathbf{k} + \cos v\,\mathbf{i} + \sin v\cos u\,\mathbf{j} + \sin v \sin u\,\mathbf{k}$$
$$= \cos v\,\mathbf{i} + \cos u(2+\sin v)\mathbf{j} + \sin u(2+\sin v)\mathbf{k}$$

$u \in [0, \pi/2]$, $v \in [0, 2\pi]$, as required.

(b) Area $= \iint_S 1 \cdot dS$, where

$$dS = \left| \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} \right| du\, dv$$

$$= \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, du\, dv.$$

We have $x = \cos v$, $y = \cos u(2+\sin v)$, $z = \sin u(2+\sin v)$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} -\sin u(2+\sin v) & \cos u \cos v \\ \cos u(2+\sin v) & \sin u \cos v \end{vmatrix} =$$

$$= -\sin^2 u \cos v(2+\sin v) - \cos^2 u \cos v(2+\sin v)$$

$$= -\cos v(2+\sin v)$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} \cos u(2+\sin v) & \sin u \cos v \\ 0 & -\sin v \end{vmatrix} = -\cos u \sin v (2+\sin v)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & -\sin v \\ -\sin u(2+\sin v) & \cos u \cos v \end{vmatrix} = -\sin u \sin v(2+\sin v).$$

$$dS = \left(\cos^2 v(2+\sin v)^2 + \cos^2 u \sin^2 v(2+\sin v)^2 + \sin^2 u \sin^2 v (2+\sin v)^2\right)^{1/2}$$

$$= \sqrt{(2+\sin v)^2} = |2+\sin v| = 2+\sin v$$

(as $-1 \le \sin v \le 1$)

So $\text{Area} = \int_0^{\pi/2} du \int_0^{2\pi} dv \, (2 + \sin v)$

$$= \frac{\pi}{2} \left(2v - \cos v \right)_0^{2\pi} = \frac{\pi}{2} \left(4\pi \right) = 2\pi^2.$$

(c) $\rho(x,y,z) = x + x\sqrt{y^2 + z^2}$ (note: negative when $x < 0$)

On S, we have $\rho(\underset{\sim}{r}(u,v)) = \cos v + \cos v \, (2 + \sin v)$

$$\iint_S \rho \, dS = \int_0^{\pi/2} du \int_0^{2\pi} dv \left(\cos v \, (2 + \sin v) + \cos v \, (2 + \sin v)^2 \right)$$

$$= \int_0^{\pi/2} du \int_0^{2\pi} dv \left[2\cos v + \underbrace{\cos v \sin v}_{\frac{1}{2}\sin 2v} + \cos v \, (2 + \sin v)^2 \right]$$

$$= \frac{\pi}{2} \cdot \left[2\sin v - \tfrac{1}{4}\cos 2v + \tfrac{1}{3}(2 + \sin v)^3 \right]_0^{2\pi}$$

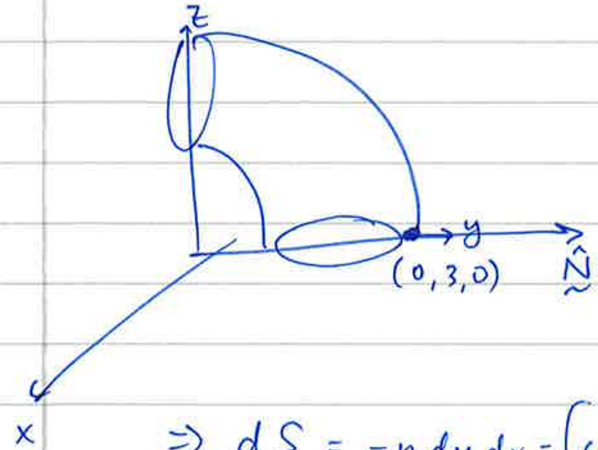
$= 0$ as expected, since $\rho(-x,y,z) = -\rho(x,y,z)$ (ρ is odd in the x-variable), but our surface S is symmetric under reflection in the yz-plane ($x \longleftrightarrow -x$).

(d) Need the outward pointing surface element $d\underset{\sim}{S} = \hat{\underset{\sim}{N}} dS$

Recall $\underset{\sim}{n} = \frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v} = \frac{\partial(y,z)}{\partial(u,v)} \underset{\sim}{i} + \frac{\partial(z,x)}{\partial(u,v)} \underset{\sim}{j} + \frac{\partial(x,y)}{\partial(u,v)} \underset{\sim}{k}$ is a normal vector to S. We have:

$$\underset{\sim}{n} = -\cos v(2+\sin v)\underset{\sim}{i} - \cos u \sin v(2+\sin v)\underset{\sim}{j} - \sin u \sin v(2+\sin v)\underset{\sim}{k}.$$

Then $d\underset{\sim}{S} = \pm \underset{\sim}{n}\, du\, dv$. We need the outward pointing normal. Check at the point $(0,3,0)$, where the outward normal must be in the $+\underset{\sim}{j}$ direction. This corresponds to $u=0$, $v=\frac{\pi}{2}$. ($\underset{\sim}{r}(0,\frac{\pi}{2}) = (0,3,0)$).



$\hat{\underset{\sim}{N}} = \underset{\sim}{j}$ here

$\underset{\sim}{n}(0,\frac{\pi}{2}) = 0\underset{\sim}{i} - 3\underset{\sim}{j} + 0\underset{\sim}{k}$ which is in the wrong direction. So we need to choose $-$ sign.

$$\Rightarrow d\underset{\sim}{S} = -\underset{\sim}{n}\, du\, dv = \Big[\cos v(2+\sin v)\underset{\sim}{i} + \cos u \sin v(2+\sin v)\underset{\sim}{j} + \sin u \sin v(2+\sin v)\underset{\sim}{k}\Big]\, du\, dv$$

(e) $\underset{\sim}{G}(\underset{\sim}{r}(u,v)) = \big[e^{+\cos v}\big(e^{\cos u(2+\sin v)} + e^{-\sin u(2+\sin v)}\big) + \cos v(\sin \ldots$ etc...

Then Flux $= \int_0^{\pi/2} du \int_0^{2\pi} \underset{\sim}{G}(\underset{\sim}{r}(u,v)) \cdot d\underset{\sim}{S}$.

I entered this integral into Mathematica. I will let you know if it finds the answer.

(numerically, I received the answer 6.28319...)

(f) $\dfrac{\partial G_1}{\partial x} = -e^{x}(e^{y} + e^{-z}) + \sin y + e^{-z}$

$\dfrac{\partial G_2}{\partial y} = e^{x-z} - \sin y$

$\dfrac{\partial G_3}{\partial z} = e^{x+y} - e^{-z}$

$\underset{\sim}{\nabla} \cdot \underset{\sim}{G} = -e^{y+x} - e^{x-z} + \sin y + e^{-z} + e^{x-z} - \sin y + e^{x+y} - e^{-z}$
$= 0.$

(g) Let S_1 be the disc in the x-y plane of radius 1, centred at $(0,2,0)$, with the unit normal $\underset{\sim}{\hat{N}}_1 = \underset{\sim}{k}$.

Let S_2 be the disc in the xz-plane of radius 1, centred at $(0,0,2)$, with $\underset{\sim}{\hat{N}}_2 = \underset{\sim}{j}$

Let $\hat{S}$ be the quarter-torus S, together with both "end caps" (the two discs mentioned above), with the outward orientation. Then $\hat{S}$ is a closed surface, and we have

$$\hat{S} = S - S_1 - S_2$$

(due to the differing orientations on S_1 and S_2 when considered as part of $\hat{S}$)

Let D be the solid volume bounded by $\hat{S}$.

By the divergence theorem,

$$\iiint_D \underset{\sim}{\nabla} \cdot \underset{\sim}{G}\, dV = \oiint_{\hat{S}} \underset{\sim}{G} \cdot \underset{\sim}{\hat{N}}\, dS = \oiint_{\hat{S}} \underset{\sim}{G} \cdot d\underset{\sim}{S}.$$

But $\underset{\sim}{\nabla} \cdot \underset{\sim}{G} = 0 \Rightarrow \oiint_{\hat{S}} \underset{\sim}{G} \cdot d\underset{\sim}{S} = 0$.

But $\oiint_{\hat{S}} = \iint_S - \iint_{S_1} - \iint_{S_2} = 0$

$$\Rightarrow \iint_S \underset{\sim}{G} \cdot d\underset{\sim}{S} = \iint_{S_1} \underset{\sim}{G} \cdot d\underset{\sim}{S} + \iint_{S_2} \underset{\sim}{G} \cdot d\underset{\sim}{S}$$

On S_1, $z = 0 \Rightarrow \underset{\sim}{G}(x,y,z) = \underset{\sim}{G}(x,y,0)$

and $\underset{\sim}{G} \cdot d\underset{\sim}{S} = \underset{\sim}{G} \cdot \quad dS = \underset{\sim}{G} \cdot \underset{\sim}{k}\, dS = G_3\, dS.$

We have $G_3(x,y,0) = 1$, so

$$\iint_{S_1} \underset{\sim}{G} \cdot d\underset{\sim}{S} = \iint_{S_1} G_3\, dS = \iint_{S_1} 1\, dS = \text{area of } S_1 = \pi$$

On S_2, $y = 0$, and $d\underset{\sim}{S} = \underset{\sim}{\hat{N}}_2\, dS = \underset{\sim}{j}\, dS \Rightarrow \underset{\sim}{G} \cdot d\underset{\sim}{S} = G_2\, dS.$

We have $G_2(x, 0, z) = 1$. Thus

$$\iint_{S_2} \underset{\sim}{G} \cdot d\underset{\sim}{S} = \iint_{S_2} 1\, dS = \text{area of } S_2 = \pi.$$

Hence, $\iint_S \underset{\sim}{G} \cdot d\underset{\sim}{S} = \pi + \pi = 2\pi.$

(h) $\underset{\sim}{F} = (-ye^{x-z} - ze^{x+y})\underset{\sim}{i} + e^{-z}(x - e^{x})\underset{\sim}{j} - (e^{x+y} + x\cos y)\underset{\sim}{k}$.

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{-}{i} & j & \underset{-}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -ye^{x-z} - ze^{x+y} & e^{-z}(x - e^{x}) & -e^{x+y} - x\cos y \end{vmatrix}$$

$$= \underset{\sim}{i}\left(-e^{x+y} + x\sin y + e^{-z}(x - e^{x})\right)$$

$$+ \underset{\sim}{j}\left(ye^{x-z} - e^{x+y} + e^{x+y} + \cos y\right)$$

$$+ \underset{\sim}{k}\left(e^{-z}(1 - e^{x}) + e^{x-z} + ze^{x+y}\right)$$

$$= G_1\underset{\sim}{i} + G_2\underset{\sim}{j} + G_3\underset{\sim}{k} = \underset{\sim}{G}(x, y, z).$$

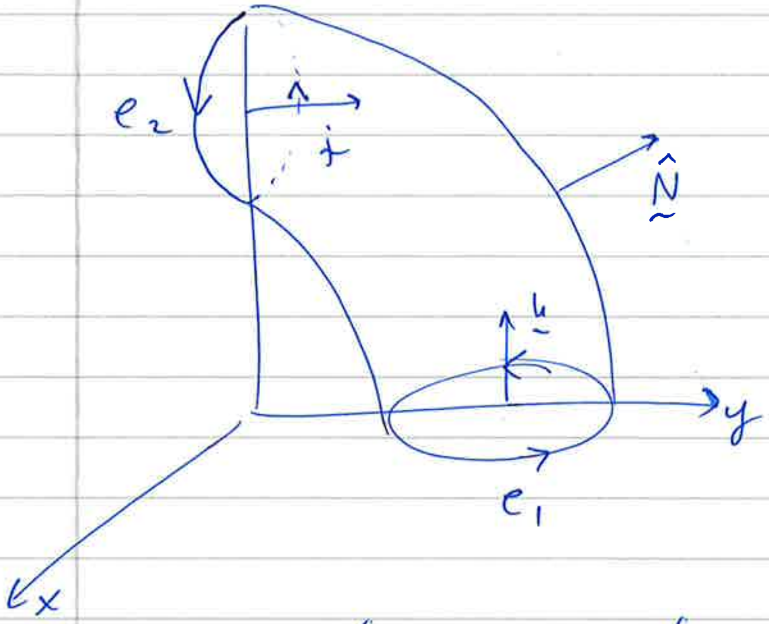
(i) By Stokes' theorem,

$$\iint_S \underset{\sim}{G} \cdot \underset{\sim}{\hat{N}}\, dS = \iint_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS = \oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

where $\mathcal{C}$ is the boundary of S with the induced orientation.

In our case, $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1$ is the boundary circle of the disc S_1 and $\mathcal{C}_2$ is the boundary circle of the disc S_2.

As can be seen on the next page, the orientations on $\mathcal{C}_1$ and $\mathcal{C}_2$ induced by $\underset{\sim}{\hat{N}}$ on S



agree with the orientations induced by $\hat{N}_1 = k$ and $\hat{N}_2 = j$, respectively.

Thus $\oint_e F \cdot dr = \oint_{e_1} F \cdot dr + \oint_{e_2} F \cdot dr$.

On e_1, $r(t) = \cos t\, i + (2 + \sin t) j$, $t \in [0, 2\pi]$.

$v(t) = -\sin t\, i + \cos t\, j$

$$F(r(t)) = -(2 + \sin t) e^{\cos t} i + (\cos t - e^{\cos t}) j + F_3 k$$

$$F \cdot v = +(2 + \sin t)\sin t\, e^{\cos t} + \cos t(\cos t - e^{\cos t}).$$

$$\oint_{e_1} F \cdot dr = \int_0^{2\pi} F \cdot v \, dt = \int_0^{2\pi} \cos^2 t + 2 \sin t\, e^{\cos t} + \sin^2 t\, e^{\cos t} - \cos t\, e^{\cos t} \, dt$$

$(\cos^2 t = \tfrac{1}{2}(\cos 2t + 1))$

$$= \tfrac{1}{2} t + \tfrac{1}{4} \sin(2t) + e^{\cos t}(-2 - \sin t) \Big|_0^{2\pi}$$

$= \pi + 0 + 0$ (all other terms periodic in t with period 2π).

to get correct orientation.

On C_2, $\underset{\sim}{r}(t) = \cos t \, \underset{\sim}{i} + (2 - \sin t)\underset{\sim}{k}$, $t \in [0, 2\pi]$.

$\underset{\sim}{v}(t) = -\sin t \, \underset{\sim}{i} - \cos t \, \underset{\sim}{k}$.

$\underset{\sim}{F}(\underset{\sim}{r}(t)) = -(2-\sin t)e^{\cos t}\underset{\sim}{i} + F_2 \underset{\sim}{j} - (e^{\cos t} + \cos t)\underset{\sim}{k}$.

$$\oint_{C_2} \underset{\sim}{F} \cdot \underset{\sim}{v} \, dt = \int_0^{2\pi} (2-\sin t)\sin t \, e^{\cos t} + \cos t \,(e^{\cos t} + \cos t) \, dt$$

$$= \frac{t}{2} + \frac{1}{4}\sin(2t) + e^{\cos t}(-2 + \sin t) \Big|_0^{2\pi}$$

$$= \pi.$$

Thus $\iint_S \underset{\sim}{G} \cdot \hat{\underset{\sim}{N}} \, dS = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \pi + \pi = 2\pi$ as expected.