14.1

1. $(x_{11}^*, y_{11}^*) = (0,1)$, $(x_{21}^*, y_{21}^*) = (1,1)$, $(x_{31}^*, y_{31}^*) = (2,1)$
$(x_{12}^*, y_{12}^*) = (0,2)$, $(x_{22}^*, y_{22}^*) = (1,2)$, $(x_{32}^*, y_{32}^*) = (2,2)$

Riemann sum $= \sum_{j=1}^{2} \sum_{i=1}^{3} f(x_{ij}^*, y_{ij}^*) \Delta x_i \Delta y_j$

where $f(x,y) = 5 - x - y$.

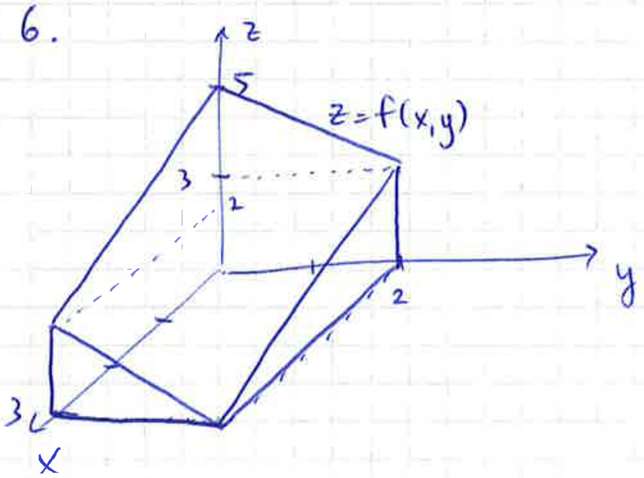
We have $\Delta x_i = 1$ for $i = 1,2,3$,
$\Delta y_j = 1$ for $j = 1,2$.

Riemann sum $= f(0,1) + f(1,1) + f(2,1) + f(0,2) + f(1,2) + f(2,2)$
$= 4 + 3 + 2 + 3 + 2 + 1 = 15$

2. Using upper right corners, Riemann sum
$= f(1,1) + f(2,1) + f(3,1) + f(1,2) + f(2,2) + f(3,2)$
$= 3 + 2 + 1 + 2 + 1 + 0 = 9$

5. Using centres of each square, Riemann sum
$= f(\tfrac{1}{2},\tfrac{1}{2}) + f(\tfrac{3}{2},\tfrac{1}{2}) + f(\tfrac{5}{2},\tfrac{1}{2}) + f(\tfrac{1}{2},\tfrac{3}{2}) + f(\tfrac{3}{2},\tfrac{3}{2}) + f(\tfrac{5}{2},\tfrac{3}{2})$.
$= 4 + 3 + 2 + 3 + 2 + 1 = 15$

6.



volume equals height at centre multiplied by base area

$= f(\tfrac{3}{2}, 1) \times 6$

$= 15$

Check: $\iint_D 5-x-y \, dA = \int_0^3 dx \int_0^2 5-x-y \, dy = \int_0^3 dx \; 5y - xy - \frac{y^2}{2}\Big|_0^2$

$= \int_0^3 10 - 2x - 2 \, dx = 8x - x^2 \Big|_0^3 = 24 - 9 = 15$ ✓

7. Define $\hat{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$

$= \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$

So we just need to count the squares whose corner closest to the origin is inside D. 24×4 such squares, so the Riemann sum $= 96$.

8. 15×4 squares have their furthest corner inside D, so Riemann sum $= 60$

10. $J = \iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$ where R is rectangle $-5 \le x \le 5$, $-5 \le y \le 5$.

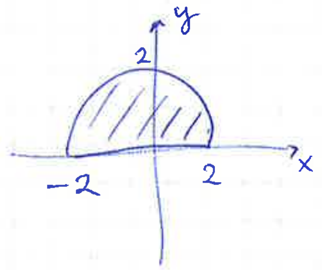
The integral gives the volume of a cylinder height 1 and base D, so $J = 1 \times \pi \times 5^2 = 25\pi \approx 78 \cdot 54$

13. $\iint_R dA$, R is rectangle $-1 \le x \le 3$, $-4 \le y \le 1$

This equals the volume of a box height 1, base R. So integral $= 1 \cdot (3-(-1)) \cdot (1-(-4))$

$= 4 \cdot 5 = 20.$

14. $\iint_D (x+3)\,dA$, $D = \{(x,y) : -2 \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}\}$

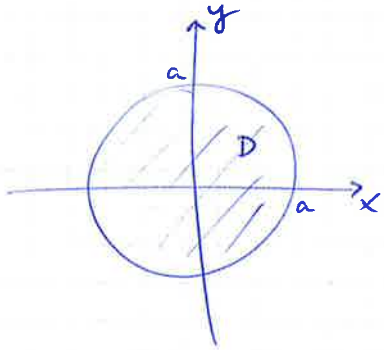


D is symmetric under reflection in x-axis, that is, when we swap x with $-x$. But then $\iint_D x\,dA = 0$, since x is positive when $x > 0$, and negative when $x < 0$, and $-x = -(x)$. (So the integral over positive x cancels the integral over negative x).

Thus $\iint_D x+3\,dA = \iint_D 3\,dA =$ volume of cylinder height 3 base D

$= 3\,\dfrac{\pi\, 2^2}{2} = 6\pi.$

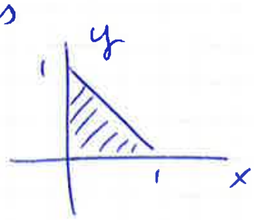
18. $\displaystyle\iint_{x^2+y^2\le a^2} \sqrt{a^2-x^2-y^2}\,dA$



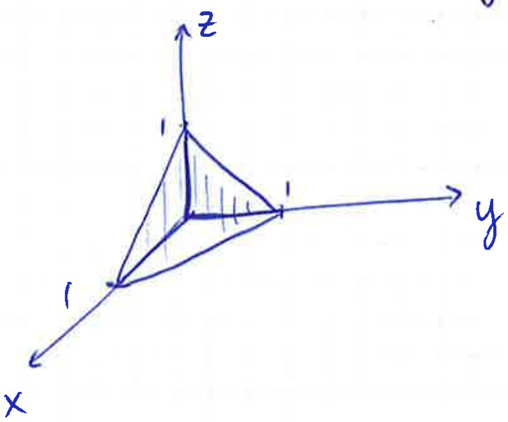
Let $z=\sqrt{a^2-x^2-y^2}$. Then

$z^2+x^2+y^2=a^2$, equation of a sphere. Since z equals the positive square root of $a^2-x^2-y^2$, the surface is the upper half of a sphere of radius a centred at O. So the integral equals the volume of the top half (hemi-sphere) of a ball $= \frac{1}{2}\cdot\frac{4}{3}\pi a^3 = \frac{4}{6}\pi a^3 = \frac{2}{3}\pi a^3$.

21. $\displaystyle\iint_T 1-x-y\,dA$, where T has vertices $(0,0),(1,0),(0,1)$.



Let $z=1-x-y$. when $x=y=0$, $z=1$.



So integral equals volume of triangular based pyramid.

$= \frac{1}{3}\times(\text{base area})\times\text{height}$

$= \frac{1}{3}\left(\frac{1}{2}\cdot 1\cdot 1\right)\cdot 1 = 1/6.$

14.2

1. $\int_0^1 dx \int_0^x (xy + y^2)\, dy = \int_0^1 dx \left(\tfrac{1}{2} x y^2 + \tfrac{1}{3} y^3 \right)\Big|_0^x$

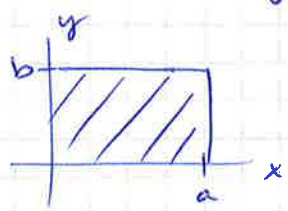
$= \int_0^1 \tfrac{1}{2} x^3 + \tfrac{1}{3} x^3 \, dx = 5/6 \quad \frac{x^4}{4}\Big|_0^1 = \frac{5}{24}$

3. $\int_0^\pi \int_{-x}^{x} \cos y \, dy\, dx = \int_0^\pi \sin y \,\Big|_{-x}^{x} \, dx$

$= \int_0^\pi \sin(x) - \sin(-x) \; dx = 2 \int_0^\pi \sin(x)\, dx$

$= -2 \cos x \,\Big|_0^\pi = 2 - (-2) = 4.$

5. $\iint_R (x^2 + y^2)\, dA$, R is $0 \le x \le a,\ 0 \le y \le b$



$= \int_0^b dy \int_0^a (x^2 + y^2)\, dx$

$= \int_0^b dy \; \tfrac{1}{3} x^3 + x y^2 \,\Big|_0^a = \int_0^b dy \left(\tfrac{1}{3} a^3 + a y^2 \right)$

$= \tfrac{1}{3} a^3 y + \tfrac{1}{3} a y^3 \,\Big|_0^b = \tfrac{1}{3} a^3 b + \tfrac{1}{3} a b^3$

$= \tfrac{1}{3} ab (a^2 + b^2)$

6. $\iint_R x^2 y^2\, dA$, R as above.

$= \int_0^a dx \int_0^b dy \; x^2 y^2 = \int_0^a dx \; \frac{1}{3} x^2 y^3 \,\Big|_0^b = \int_0^a dx \; \tfrac{1}{3} x^2 b^3$

$= \tfrac{1}{9} x^3 b^3 \,\Big|_0^a = \tfrac{1}{9} a^3 b^3$

7. $\iint_S \sin x + \cos y \, dA$, S is square $0 \le x \le \pi/2$
$0 \le y \le \pi/2$

$$= \int_0^{\pi/2} \left(\int_0^{\pi/2} \sin x + \cos y \, dx \right) dy$$

$$= \int_0^{\pi/2} -\cos x + x \cos y \Big|_0^{\pi/2} dy = \int_0^{\pi/2} 1 + \pi/2 \cos y \, dy$$

$$= y + \pi/2 \sin y \Big|_0^{\pi/2} = \pi$$