16.4 exercises

(10) $\phi = xy + z^2$. $\quad \underset{\sim}{F} = \underset{\sim}{\nabla}\phi$, then

$$\oiint_R \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV,$$

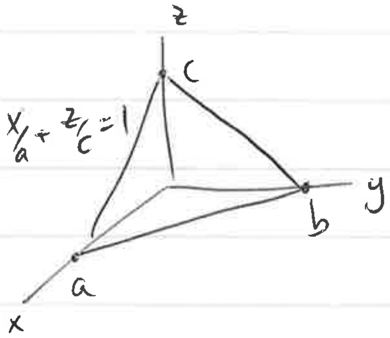
where T is tetrahedron vertices at $(0,0,0)$, $(a,0,0)$, $(0,0,b)$, $(0,0,c)$

and R is the total surface of T.

Now $\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla}\phi) = \nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} = 2$

So $\displaystyle\iiint_T \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = 2 \cdot \mathrm{vol}(T) = 2\left(\tfrac{1}{3} \cdot \tfrac{1}{2}\, a \cdot b \cdot c\right) = \frac{abc}{3}$

$$\oiint_R \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iint_{3 \text{ sides}} \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS + \iint_{S} \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS.$$



$\underset{\sim}{F} = \underset{\sim}{\nabla}\phi = y\underset{\sim}{i} + x\underset{\sim}{j} + 2z\underset{\sim}{k}$

bottom: $\underset{\sim}{\hat{N}} = -\underset{\sim}{k}$, $\quad \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}} = -2z = 0$ since $z = 0$

left: $\underset{\sim}{\hat{N}} = -\underset{\sim}{j}$, $\quad \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}} = -x$

$$\iint_{\text{left}} \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = -\int_0^a dx \int_0^{c - cx/a} x\, dz$$

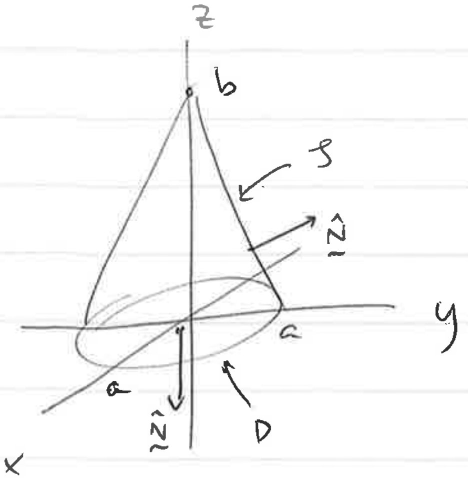
$$= -\int_0^a dx \left(xc - \frac{cx^2}{a}\right) = \int_0^a \frac{cx^2}{a} - cx\, dx = \frac{ca^2}{3} - \frac{ca^2}{2} = -\frac{ca^2}{6}$$

back: $\underset{\sim}{\hat{N}} = -\underset{\sim}{i} \Rightarrow \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}} = -y$

$$\iint_{\text{back}} \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = -\int_0^b dy \int_0^{c - cy/b} y\, dz = -\frac{cb^2}{6}$$

$$\Rightarrow \iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \frac{abc}{3} + \frac{ca^2}{6} + \frac{cb^2}{6} = \frac{c}{6}\left(2ab + a^2 + b^2\right) = \frac{c\,(a+b)^2}{6}$$

(ii)



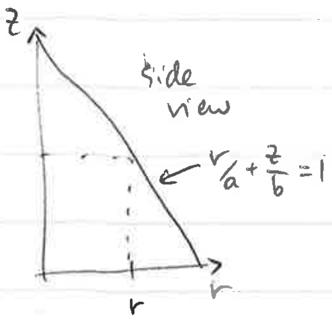
$$\underset{\sim}{F} = (x+y^2)\,\underset{\sim}{i} + (3x^2y + y^3 - x^3)\,\underset{\sim}{j} + (z+1)\,\underset{\sim}{k}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{F} = 1 + 3x^2 + 3y^2 + 1 = 2 + 3x^2 + 3y^2$$

$$\iiint_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\,dV = \iint_S \underset{\sim}{F}\cdot\underset{\sim}{\hat N}\,dS + \iint_D \underset{\sim}{F}\cdot\underset{\sim}{\hat N}\,dS$$

by divergence theorem.

In cylindrical coordinates, V has description



$$0 \le \theta \le 2\pi$$
$$0 \le r \le a$$
$$0 \le z \le b - \tfrac{b}{a} r$$

$$\iiint_V \underset{\sim}{\nabla}\cdot\underset{\sim}{F}\,dV = \int_0^{2\pi} d\theta \int_0^a dr \int_0^{b-\frac{b}{a}r} (2 + 3r^2)\, r\, dz$$

$$= 2\pi \int_0^a 2r\left(b - \tfrac{b}{a} r\right) + 3r^3\left(b - \tfrac{b}{a} r\right)\, dr$$

$$= 2\pi \int_0^a 2rb - 2\tfrac{b}{a} r^2 + 3r^3 b - 3\tfrac{b}{a} r^4\, dr$$

$$= 2\pi \left[r^2 b - \frac{2b}{3a} r^3 + 3/4\, r^4 b - \frac{3b}{5a} r^5 \right]_0^a$$

$$= 2\pi \left[a^2 b - 2b/3\, a^2 + 3/4\, a^4 b - \frac{3\,b\,a^4}{5} \right] = 2\pi\left[\tfrac{1}{3} a^2 b + \tfrac{3}{20} b a^4 \right]$$

$$= \tfrac{2}{3}\pi a^2 b + \frac{3}{10}\pi a^4 b.$$

On D, $\hat{N} = -k$, so $F \cdot \hat{N} = -z-1$.

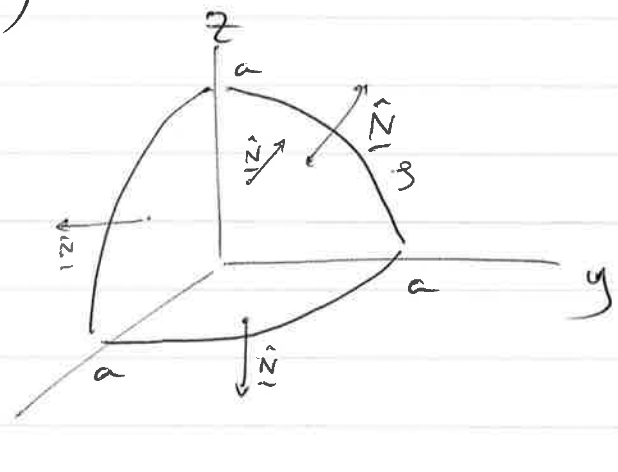
But $z=0$ on D so $F \cdot \hat{N} = -1$.

Thus $\iint_D F \cdot \hat{N}\, dS = -\text{area}(D) = -\pi a^2$.

So $\iint_S F \cdot \hat{N}\, dS = \iiint_V \nabla \cdot F\, dV - \iint_D F \cdot \hat{N}\, dS$

$$= \tfrac{2}{3}\pi a^2 b + \tfrac{3}{10}\pi a^4 b + \pi a^2$$

(12)



$F = (y+xz)i + (y+yz)j - (2x+z^2)k$

$\nabla \cdot F = z + 1 + z - 2z = 1$

On bottom, $\hat{N} = -k$, $F \cdot \hat{N} = +(2x+z^2) = 2x \quad (z=0)$

$$\iint_{\text{bottom}} F \cdot \hat{N}\, dS = 2\int_0^{\pi/2} d\theta \int_0^a r\cos\theta \, r\, dr = 2\int_0^{\pi/2} \cos\theta\, d\theta \int_0^a r^2\, dr$$

$$= 2 \sin\theta \Big|_0^{\pi/2} \cdot \tfrac{1}{3} r^3 \Big|_0^a = \tfrac{2}{3} \cdot 1 \cdot a^3 = \frac{2a^3}{3}$$

On left face, $\hat{\underset{\sim}{N}} = -\underset{\sim}{j} \Rightarrow \quad \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -(y + yz) = 0 \quad (y=0)$

On back face, $\hat{\underset{\sim}{N}} = -\underset{\sim}{i} \Rightarrow \quad \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -(y + xz) = -y \quad (x=0)$

$$\iint_{\text{back}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS = -\int_0^{\pi/2} d\theta \int_0^a r\cos\theta \, r \, dr \qquad \left(\begin{array}{l} y = r\cos\theta \\ z = r\sin\theta \end{array} \right)$$

$$= -a^3/3 .$$

let D = domain bounded by S in first octant

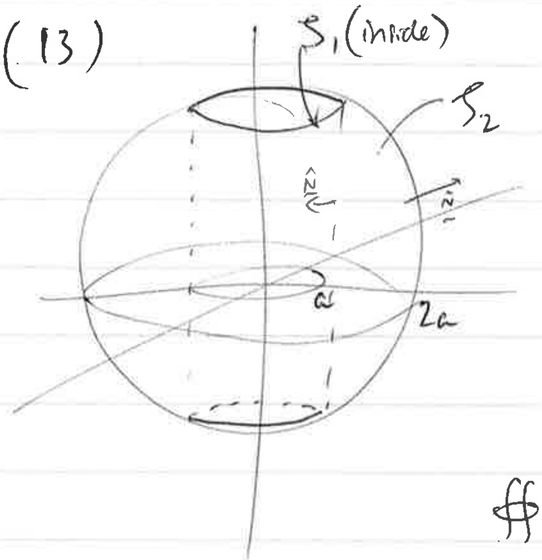
$$\iiint_D \underbrace{\underset{\sim}{\nabla} \cdot \underset{\sim}{F}}_{=1} \, dV = \text{vol}(D) = \tfrac{1}{8} \tfrac{4}{3} \pi a^3 = \frac{1}{6} \pi a^3$$

By divergence theorem,

$$\iiint_D \underset{\sim}{\nabla} \cdot \underset{\sim}{F} \, dV = \oiint_{\substack{\text{boundary} \\ \text{surface} \\ \text{of } D}} \underset{\sim}{} \cdot \hat{\underset{\sim}{N}} \, dS = \left(\iint_S + \iint_{\text{bottom}} + \iint_{\text{left}} + \iint_{\text{back}} \right) \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS$$

$$\Rightarrow \iint_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS = \pi a^3/6 - \frac{2a^3}{3} + a^3/3 = \frac{(\pi - 2) a^3}{6}$$

(13)



Sphere with cylindrical hole drilled out.

(a) $\underline{F} = (x+yz)\underline{i} + (y-xz)\underline{j} + (z - e^x \sin y)\underline{k}$

$\underline{\nabla}\cdot\underline{F} = 1 + 1 + 1 = 3.$

$$\oiint_{S_1+S_2} \underline{F}\cdot\underline{\hat{N}}\,dS = \iiint_D \underline{\nabla}\cdot\underline{F}\,dV = 3\,\mathrm{vol}(D).$$

Compute volume of <u>top half</u> of cylinder with curved cap using cylindrical coords:

$0 \le \theta \le 2\pi$

$0 \le r \le a$

$0 \le z \le \sqrt{4a^2 - r^2}$

$$\mathrm{Vol} = \int_0^{2\pi} d\theta \int_0^a dr \int_0^{\sqrt{4a^2-r^2}} r\,dz = 2\pi \int_0^a r\sqrt{4a^2-r^2}\,dr$$

$$= 2\pi \tfrac{2}{3}\left(-\tfrac{1}{2}\right)(4a^2-r^2)^{3/2}\Big|_0^a = -\tfrac{2\pi}{3}\left[(3a^2)^{3/2} - (4a^2)^{3/2}\right]$$

$$= +\tfrac{2\pi}{3}\left(8a^3 - (\sqrt{3})^3 a^3\right) = \underline{\frac{2\pi a^3}{3}}(8 - 3\sqrt{3}) \quad (>0)$$

total cylinder with curved caps $= \dfrac{4\pi a^3}{3}(8 - 3\sqrt{3})$.

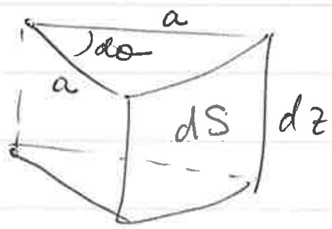
So $\mathrm{vol}(D) = \tfrac{4}{3}\pi(2a)^3 - \dfrac{4\pi a^3(8-3\sqrt{3})}{3} = 4\sqrt{3}\pi a^3$

$$\therefore \oiint_{S_1+S_2} \underline{F}\cdot\underline{\hat{N}}\,dS = 3\cdot 4\sqrt{3}\pi a^3 = 12\sqrt{3}\pi a^3.$$

(b) S_1 is the curved surface of a cylinder, radius a, height $2\sqrt{3}a$.

We have that $\hat{\underset{\sim}{N}} = \frac{-(x\underset{\sim}{i} + y\underset{\sim}{j})}{\sqrt{x^2+y^2}} = \frac{-x\underset{\sim}{i} - y\underset{\sim}{j}}{a} =$

And $dS = a \; d\theta \, dz$



So $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS = \frac{1}{a}(-x^2 - xyz - y^2 + xyz) \cdot a \, d\theta \, dz$

$= -r^2 \, d\theta \, dz = -a^2 \, d\theta \, dz$ on S_1.

Then $\iint_{S_1} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} \, dS = \int_0^{2\pi} d\theta \int_{-\sqrt{3}a}^{\sqrt{3}a} -a^2 \, dz$

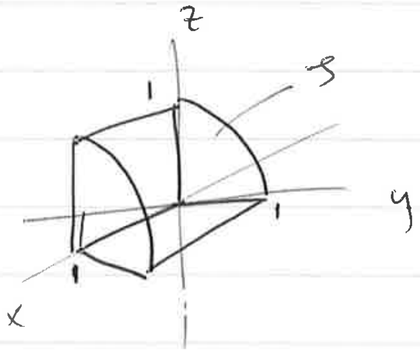
$= -a^2 \cdot 2\pi \cdot 2\sqrt{3}a = -4\sqrt{3}\pi a^3$

(c) $\iint_{S_2} = 3\,\mathrm{vol}(D) - \iint_{S_1} = 12\sqrt{3}\pi a^3 + 4\sqrt{3}\pi a^3 = 16\sqrt{3}\pi a^3.$

14. $\underset{\sim}{F} = 3xz^2\,\underset{\sim}{i} - x\,\underset{\sim}{j} - y\,\underset{\sim}{k}$

$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 3z^2$

Let $y = r\cos\theta$, $z = r\sin\theta$, $x = x$.
$\Rightarrow dV = r\,dr\,d\theta\,dx$



Orientation on S not specified, so assume upwards-pointing normal.

$$\underset{\text{boundary of } D}{\oint\!\!\oint} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = \iiint_D \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV$$

$$= 3 \int_0^{\pi/2} d\theta \int_0^1 dr \int_0^1 r^2 \sin^2\theta \cdot r\, dx$$

$$= 3 \int_0^{\pi/2} \tfrac{1}{2}[1 - \cos 2\theta]\, d\theta \int_0^1 r^3\, dr \int_0^1 dx$$

$$= \tfrac{3}{2} \cdot \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} \cdot \left.\frac{r^4}{4}\right|_0^1 \cdot 1$$

$$= \tfrac{3}{2}\left[\tfrac{\pi}{2}\right] \cdot \tfrac{1}{4} = \frac{3\pi}{16}.$$

back face: $\hat{\underset{\sim}{N}} = -\underset{\sim}{i}$, $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = -3xz^2 = 0 \quad (x = 0)$
front face: $\hat{\underset{\sim}{N}} = \underset{\sim}{i}$, $\underset{\sim}{F} \cdot \hat{\underset{\sim}{N}} = 3xz^2 = 3z^2 \quad (x = 1)$

$$\iint_{\text{front}} \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = 3 \int_0^{\pi/2} d\theta \int_0^1 r^2 \sin^2\theta\, r\, dr$$

$$= \tfrac{3}{2} \cdot \left[\tfrac{\pi}{2}\right] \cdot \tfrac{1}{4} = \frac{3\pi}{16}$$

left face: $\hat{N} = -j \Rightarrow F \cdot \hat{N} = x$

$$\iint_{\substack{\text{left}\\\text{face}}} F \cdot \hat{N}\, dS = \int_0^1 dx \int_0^1 x\, dz = \int_0^1 x\, dx = \tfrac{1}{2}$$

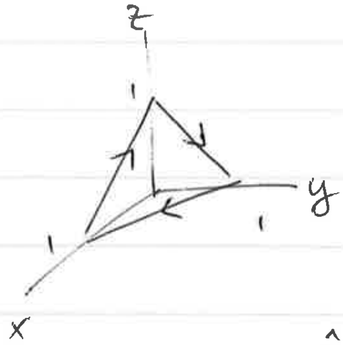
bottom face: $\hat{N} = -k \Rightarrow F \cdot \hat{N} = y$

$$\iint_{\text{bottom}} F \cdot \hat{N}\, dS = \int_0^1 dx \int_0^1 y\, dy = \tfrac{1}{2}$$

Thus $$\iint_S F \cdot \hat{N}\, dS = \iiint_D \nabla \cdot F\, dV - \left(\iint_{\text{back}} + \iint_{\text{front}} + \iint_{\text{bottom}} + \iint_{\text{left}} \right) F \cdot \hat{N}\, dS$$

$$= \tfrac{3\pi}{16} - 0 - \tfrac{3\pi}{16} - \tfrac{1}{2} - \tfrac{1}{2} = -1.$$

16.5 (1)



$$\oint_C xy\,dx + yz\,dy + zx\,dz$$

$$= \oint_C (xy\,\underset{\sim}{i} + yz\,\underset{\sim}{j} + xz\,\underset{\sim}{k}) \cdot d\underset{\sim}{r} = \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$$

$$= \iint_T (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS \quad \text{by Stokes' theorem,}$$

where $\hat{\underset{\sim}{N}}$ is unit normal to T pointing towards origin. T is part of plane $x+y+z=1$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix} = \underset{\sim}{i}(-y) + \underset{\sim}{j}(-z) + \underset{\sim}{k}(-x)$$

$$= -(\underset{\sim}{i}\,y + \underset{\sim}{j}\,z + \underset{\sim}{k}\,x).$$

By symmetry it is clear that $\hat{\underset{\sim}{N}} = -\frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

So $(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} = \frac{1}{\sqrt{3}}(x+y+z)$. Let $G(x,y,z) = x+y+z$

$\Rightarrow T:\ G(x,y,z) = 1$

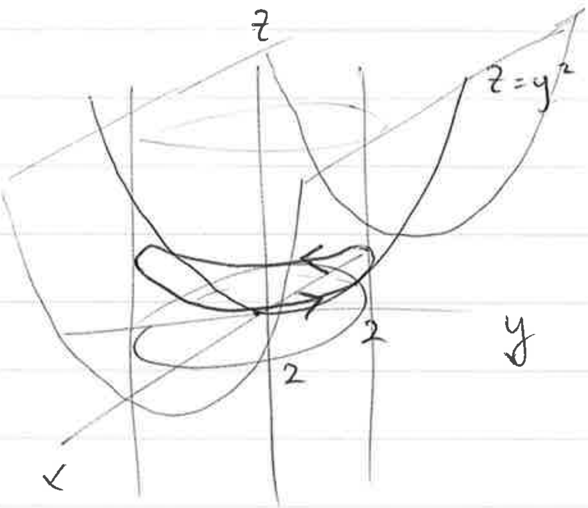
$$dS = \frac{|\underset{\sim}{\nabla} G|}{\left|\frac{\partial G}{\partial z}\right|}\,dx\,dy = \frac{\sqrt{3}}{1}\,dx\,dy$$

Let R: triangle in xy-plane, $0 \le x \le 1$, $0 \le y \le 1-x$.

$$\Rightarrow \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_R x+y+z\ dx\,dy = \iint_R dx\,dy = \text{area}(R) = \tfrac{1}{2}.$$

(2) $\oint_{\mathcal{C}} y\,dx - x\,dy + z^2\,dz = \oint_{\mathcal{C}} (\underline{i}\,y - \underline{j}\,x + \underline{k}\,z^2)\cdot d\underline{r} = \oint_{\mathcal{C}} \underline{F}\cdot d\underline{r}$

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & z^2 \end{vmatrix} = \underline{i}\,(0) + \underline{j}\,(0) + \underline{k}\,(-2) = -2\underline{k}$$



$$\oint_{\mathcal{C}} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \iint_{S} (\underline{\nabla}\times\underline{F})\cdot\hat{\underset{\sim}{N}}\,dS.$$

we take S to be the piece of $z = y^2$ above the disc D $x^2 + y^2 \leq 4$ in x-y plane.

Then $z = f(x,y) = y^2$, $\partial f/\partial x = 0$, $\partial f/\partial y = 2y$

$$\Rightarrow \hat{\underset{\sim}{N}}\,dS = d\underset{\sim}{S} = \pm\left(-\partial f/\partial x\,\underline{i} - \partial f/\partial y\,\underline{j} + \underline{k}\right)dx\,dy$$

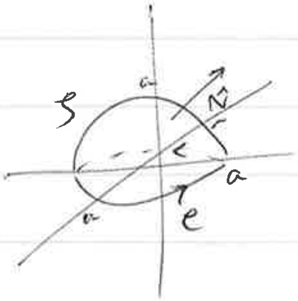
$$= +\left(-2y\,\underline{j} + \underline{k}\right)dx\,dy$$

where we chose '+' to get the correct induced orientation on $\mathcal{C}$.

$$(\underline{\nabla}\times\underline{F})\cdot\hat{\underline{N}}\,dS = -2\,dx\,dy$$

$$\Rightarrow \oint_{\mathcal{C}} \underline{F}\cdot d\underline{r} = -2\iint_{D} dx\,dy = -2\cdot\pi\cdot 2^2 = -8\pi.$$

(3) $$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



$$\mathbf{F} = 3y\,\mathbf{i} - 2xz\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$$

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$$

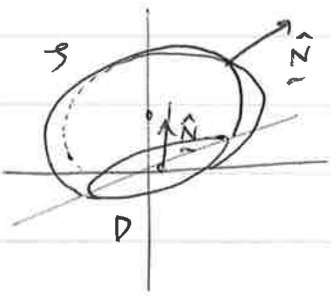
$$d\mathbf{r} = (-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j})\,dt$$

$$\mathbf{F} \cdot d\mathbf{r} = 3a\sin t\,(-a\sin t)\,dt \qquad (z=0)$$
$$= -3a^2 \sin^2 t\,dt$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -3a^2 \int_0^{2\pi} \sin^2 t\,dt = -\frac{3a^2}{2}\int_0^{2\pi} 1 - \cos 2t\,dt$$

$$= -3\pi a^2.$$

(4) $$x^2 + y^2 + 2(z-1)^2 = 6, \quad z \geq 0.$$



$$\mathbf{F} = (xz - y^3\cos z)\,\mathbf{i} + x^3 e^z\,\mathbf{j} + xyz\,e^{x^2+y^2+z^2}\,\mathbf{k}$$

Let R be enclosed ~~volume~~ region by S, D.
Note $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, (always true), so

$$\iiint_R \nabla \cdot (\nabla \times \mathbf{F})\,dV = 0$$

By divergence theorem, $$\iiint_R \nabla \cdot (\nabla \times \mathbf{F})\,dV = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} - \iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

when D has $\mathbf{N} = \mathbf{k}$ as shown.

So $\iint_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{\hat{N}}\, dS = \iint_D (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{N}}\, dS$

On D, $\underset{\sim}{\hat{N}} = \underline{k}$, so we only need $\underline{k}$ component of $\underset{\sim}{\nabla} \times \underset{\sim}{F}$: $\quad \underline{k} \cdot (\underset{\sim}{\nabla} \times \underset{\sim}{F}) = \partial/\partial x\,(x^3 e^z) - \partial/\partial y\,(xz - y^3 \cos z)$

$= 3x^2 e^z + 3y^2 \cos z = 3(x^2 + y^2) \quad (z = 0 \text{ on } D).$

Also, S intersects $z=0$ in $x^2 + y^2 + 2(-1)^2 = 6$

$\Rightarrow x^2 + y^2 = 4$ circle, $r = 2$.

So $D: x^2 + y^2 \leq 4, \quad z = 0.$

Thus $\iint_D (\underline{\nabla} \times \underline{F}) \cdot \underset{\sim}{\hat{N}}\, dS = 3 \int_0^{2\pi} d\theta \int_0^2 r^2 \cdot r\, dr = 6\pi \cdot \frac{16}{4} = 24\pi.$

(5) $\mathcal{C}: \quad x^2 + y^2 + z^2 = a^2$ intersected with $x + y + z = 0$.

$\oint_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r}, \quad \underline{F} = y\underline{i} + z\underline{j} + x\underline{k}$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \underline{i}(-1) + \underline{j}(-1) + \underline{k}(-1)$$

$$= -(\underline{i} + \underline{j} + \underline{k}).$$

Stokes $\Rightarrow \oint_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \iint_D (\underline{\nabla} \times \underline{F}) \cdot \underset{\sim}{\hat{N}}\, dS$ where D is disc bounded by $\mathcal{C}$.

Let $G(x, y, z) = x + y + z = 0$ on S, then $\underset{\sim}{N} = \pm \underset{\sim}{\nabla} G = \pm(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}) \Rightarrow \hat{\underset{\sim}{N}} = \pm \frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

$\Rightarrow (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} = \pm \sqrt{3}$

$\Rightarrow \oint_{C} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \pm \iint_{D} \sqrt{3}\, dS = \pm\sqrt{3}\ \text{area}(D) = \pm \sqrt{3}\pi a^2$

If we choose $\hat{\underset{\sim}{N}} = -\frac{1}{\sqrt{3}}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$

then $(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}} = +\sqrt{3}$ and we get

$\oint_{C} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \sqrt{3}\pi a^2.$

(6) $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$, $\underset{\sim}{r} = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + \underbrace{\sin 2t}_{2\sin t \cos t}\, \underset{\sim}{k}$, $0 \le t \le 2\pi$.

Note that $\underset{\sim}{r}$ is the graph of $f(x,y) = 2xy$ over the circle $x^2 + y^2 = 1$ in xy-plane.

$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_S (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS$ where $\hat{\underset{\sim}{N}}$ is pointing up.

$\underset{\sim}{F} = (e^x - y^3)\underset{\sim}{i} + (e^y + x^3)\underset{\sim}{j} + e^z \underset{\sim}{k}$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x - y^3 & e^y + x^3 & e^z \end{vmatrix} = \underset{\sim}{i}(0) + \underset{\sim}{j}(0 - 0) + \underset{\sim}{k}(3x^2 + 3y^2)$$

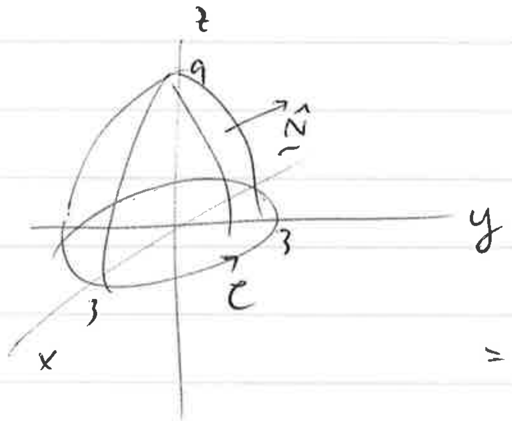
$$\frac{\partial f}{\partial x} = 2y \qquad \frac{\partial f}{\partial y} = 2x \quad \Rightarrow \quad \hat{\underset{\sim}{N}}\, dS = (-2y\,\underset{\sim}{i} - 2x\,\underset{\sim}{j} + \underset{\sim}{k})\, dx\, dy$$

$$(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS = (3x^2 + 3y^2)\, dx\, dy$$

$$\Rightarrow \oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D 3r^2\, dx\, dy \quad \text{where } D: x^2+y^2 \le 1,\ z=0.$$

$$= 3 \int_0^{2\pi} d\theta \int_0^1 r^3\, dr = \tfrac{3}{4} \cdot 2\pi = \frac{3\pi}{2}.$$

(7) $\underset{\sim}{F} = -y\,\underset{\sim}{i} + x^2\,\underset{\sim}{j} + z\,\underset{\sim}{k}$, C: $x = 3\cos\theta$ $y = 3\sin\theta$, $z=0$



$$\underset{\sim}{F} \cdot d\underset{\sim}{r} = (-3\sin\theta \cdot (-3\sin\theta) + 9\cos^2\theta \cdot 3\cos\theta)\, d\theta$$

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = 9\int_0^{2\pi} \sin^2\theta + 3\cos^3\theta \quad d\theta$$

$$= 9 \int_0^{2\pi} \tfrac{1}{2} - \underbrace{\tfrac{1}{2}\cos 2\theta}_{\text{integrates to } 0} + 3\cos\theta(1-\sin^2\theta)\, d\theta$$

$= 9\pi$ (all other terms integrate to 0).

OR: use Stokes' theorem $\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D (\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \hat{\underset{\sim}{N}}\, dS$

where D is disc $x^2+y^2 \le 9$, $z=0$, with $\hat{\underset{\sim}{N}} = \underset{\sim}{k}$.

Then $(\underset{\sim}{\nabla} \times \underset{\sim}{F}) \cdot \underset{\sim}{k} = \partial/\partial x\,(x^2) - \partial/\partial y\,(-y) = 2x + 1$

$$\oint_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_D \underbrace{2x}_{\substack{\text{integrates to}\\ \text{zero due to}\\ \text{symmetry}}} + 1\, dS = \text{area}(D) = 9\pi.$$

8. $\oint_C \underline{F} \cdot d\underline{r}$, $\quad \underline{F} = ye^x \underline{i} + (x^2 + e^x)\underline{j} + z^2 e^z \underline{k}$

$\underline{r}(t) = (1 + \cos t)\underline{i} + (1 + \sin t)\underline{j} + (1 - \cos t - \sin t)\underline{k}$

$0 \le t \le 2\pi$.

$= x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

we have $3 - x - y = 3 - 1 - \cos t - 1 - \sin t = z$

$\Rightarrow x + y + z = 3$ on $\underline{r}$,

so $\underline{r}$ lies in plane $x + y + z = 3$.

The projection of $\underline{r}$ into xy-plane is the circle $x = 1 + \cos t$, $y = 1 + \sin t$, centred at $(1,1)$ with radius 1.

Let $D = \{(x-1)^2 + (y-1)^2 \le 1\}$, so that the boundary of D is the circle $(x-1)^2 + (y-1)^2 = 1$, the projection of C onto the xy-plane.

Thus D is the projection of a surface S in the plane $x+y+z=3$ with boundary C.

Need to choose $\hat{\underline{N}}$ on S with positive z-component, so that the induced orientation on C agrees with the orientation given by the parametrisation $\underline{r}(t)$ (anticlockwise when viewed from above).

Let $G(x,y,z) = x + y + z$, then S is contained in the level surface $G(x,y,z) = 3$.

$\underset{\sim}{\nabla} G = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\frac{\partial G}{\partial z} = 1$

$$\Rightarrow \quad \hat{\underset{\sim}{N}}\, dS = \pm \frac{\underline{\nabla} G}{\frac{\partial G}{\partial z}}\, dx\,dy = \pm (\underline{i} + \underline{j} + \underline{k})\, dx\, dy$$

Choose '+' sign to get positive z-component.

$$\hat{\underset{\sim}{N}}\, dS = (\underline{i} + \underline{j} + \underline{k})\, dx\, dy$$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^x & x^2+e^x & z^2e^z \end{vmatrix} = \underset{\sim}{i}(0-0) + \underline{j}(0-0) + \underline{k}(2x + e^x - e^x)$$

$$= 2x\underset{\sim}{k}$$

Stokes': $\oint_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \hat{\underline{N}}\, dS = \iint_D 2x\underline{k} \cdot (\underline{i} + \underline{j} + \underline{k})\, dx\,dy$

$$= \iint_D 2x\, dx\,dy = \int_0^{2\pi} d\theta \int_0^1 2(1 + r\cos\theta)\, r\, dr$$

$$= \int_0^{2\pi} 1 + \tfrac{2}{3}\cos\theta\, d\theta = 2\pi.$$