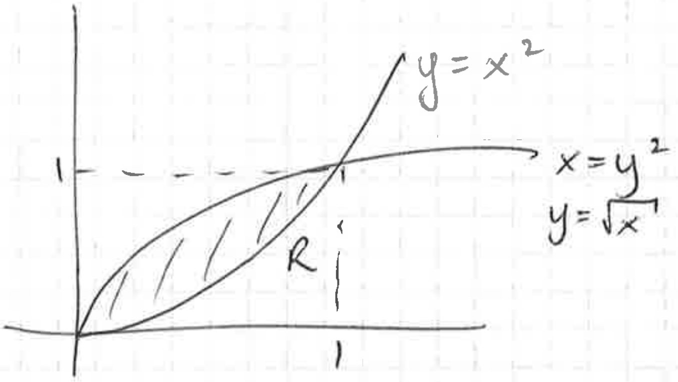
14.2

9. $\iint_R xy^2\,dA$, R in first quadrant between $y=x^2$, $x=y^2$.



Intersect when

$y = x^2 = (y^2)^2 = y^4$

$\Rightarrow y^4 - y = 0 \Rightarrow y(y^3-1) = 0$

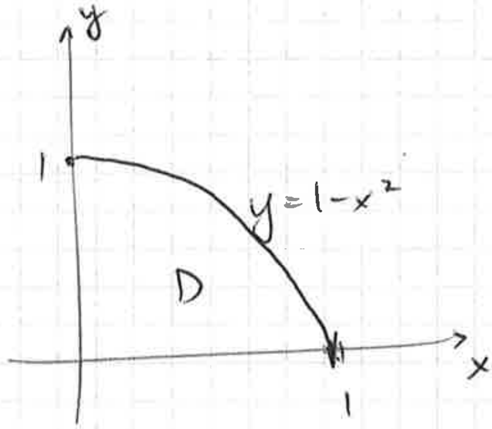
$\Rightarrow y = 0$ or $y = 1$

$\Rightarrow x = 0$ or $x = 1$

$$\int_0^1 dx \int_{x^2}^{\sqrt{x}} dy\,(xy^2) = \frac{1}{3}\int_0^1 dx\,\left(xy^3\right)\Big|_{x^2}^{\sqrt{x}} = \frac{1}{3}\int_0^1 dx\,(x^{5/2} - x^7)$$

$$= \frac{1}{3}\left[\frac{2}{7}x^{7/2} - \frac{1}{8}x^8\right]_0^1 = \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{9}{168}$$

10. $\iint_D x\cos y\,dA$, D is region in first quadrant bounded by axes and $y = 1-x^2$.



$$\int_0^1 \int_0^{1-x^2} x\cos y\,dy\,dx$$

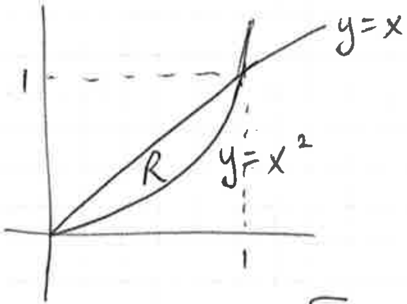
$$= \int_0^1 x\sin y\Big|_0^{1-x^2} dx$$

$$= \int_0^1 x\sin(1-x^2)\,dx$$

$$= +\frac{1}{2}\cos(1-x^2)\Big|_0^1 = \frac{1}{2}(\cos 0 - \cos 1)$$

$$= \frac{1}{2}(1 - \cos 1)$$

13 $\iint_R \frac{x}{y} e^y$, $\quad R = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le x\}$



$$\int_0^1 dx \int_{x^2}^{x} dy \left(\frac{x}{y} e^y\right)$$ seems difficult, try iterating the other way:

$$y = x^2 \iff x = \sqrt{y}$$

$$\int_0^1 dy \int_y^{\sqrt{y}} dx \left(\frac{x}{y} e^y\right) = \frac{1}{2}\int_0^1 dy \left(\frac{e^y}{y} x^2\right)\Big|_y^{\sqrt{y}}$$

$$= \frac{1}{2}\int_0^1 dy \left(\frac{e^y}{y}\cdot y - \frac{y^2}{y} e^y\right) = \frac{1}{2}\int_0^1 dy \left(e^y - y e^y\right)$$

Now let $u = y$, $v = e^y$, then $\int y e^y\, dy = \int u\, dv = uv - \int v\, du$

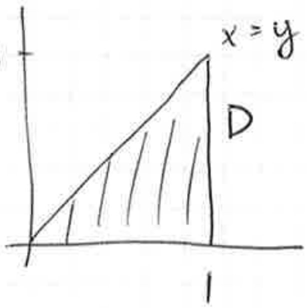
$du = dy$, $dv = e^y\, dy$

$$= y e^y - \int e^y\, dy$$

$$= y e^y - e^y$$

$$= \frac{1}{2}\left[e^y - y e^y + e^y\right]_0^1 = \frac{1}{2}(2e - e - 2) = \frac{1}{2}(e - 2)$$

15/ $\int_0^1 dy \int_y^1 e^{-x^2} dx$ cannot evaluate, switch order:



$$\int_0^1 dx \int_0^x e^{-x^2} dy = \int_0^1 dx \left(e^{-x^2} y\right)\Big|_0^x$$

$$= \int_0^1 x e^{-x^2} dx = -\frac{1}{2} e^{-x^2}\Big|_0^1$$

$$= -\frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

19. Under $z = 1 - x^2$, above $0 \le x \le 1$, $0 \le y \le x$.

$$V = \int_0^1 dx \int_0^x dy\,(1 - x^2) = \int_0^1 dx\, (y + yx^2)\Big|_0^x = \int_0^1 dx\,(x - x^3)$$

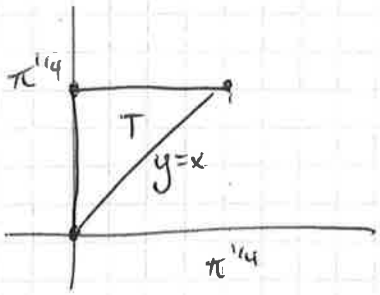
$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$$

20. Under $z = 1 - x^2$, above $0 \le y \le 1$, $0 \le x \le y$.

$$V = \int_0^1 dy \int_0^y dx\,(1 - x^2) = \int_0^1 dy\, \left(x - \tfrac{1}{3}x^3\right)\Big|_0^y$$

$$= \int_0^1 y - \tfrac{1}{3}y^3\, dy = \tfrac{1}{2}y^2 - \tfrac{1}{12}y^4 \Big|_0^1 = \tfrac{1}{2} - \tfrac{1}{12} = \tfrac{5}{12}$$

24. Under $z = x^2 \sin(y^4)$ and above triangle with vertices $(0,0)$, $(0, \pi^{1/4})$, $(\pi^{1/4}, \pi^{1/4})$.



$$V = \int_0^{\pi^{1/4}} dy \int_0^y dx\,\left(x^2 \sin(y^4)\right)$$

$$= \frac{1}{3}\int_0^{\pi^{1/4}} dy\ x^3 \sin(y^4)\Big|_0^y = \tfrac{1}{3}\int_0^{\pi^{1/4}} dy\left(y^3 \sin(y^4)\right)$$

$$= -\frac{1}{12}\cos(y^4)\Big|_0^{\pi^{1/4}} = -\tfrac{1}{12}\Big[\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}\Big]$$

$$= \tfrac{1}{6}$$

10.2

1/ $A = (-1, 2)$, $B = (2, 0)$, $C = (1, -3)$, $D = (0, 4)$.

(a) $\overrightarrow{AB} = (2-(-1), 0-2) = (3, -2) = 3\underline{i} - 2\underline{j}$

(b) $\overrightarrow{BA} = -\overrightarrow{AB} = -3\underline{i} + 2\underline{j}$

(c) $\overrightarrow{AC} = (1-(-1), -3-2) = (2, -5) = 2\underline{i} - 5\underline{j}$

(d) $\overrightarrow{BD} = (0-2, 4-0) = -2\underline{i} + 4\underline{j}$

(e) $\overrightarrow{DA} = -\overrightarrow{AD} = -(\overrightarrow{AB} + \overrightarrow{BD}) = -(3\underline{i} - 2\underline{j} \cancel{+} - 2\underline{i} + 4\underline{j})$

$= -\underline{i} - 2\underline{j}$

(f) $\overrightarrow{AB} - \overrightarrow{BC} = 3\underline{i} - 2\underline{j} - (-\underline{i} - 3\underline{j}) = 4\underline{i} + \underline{j}$

(g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD} = 2\underline{i} - 5\underline{j} - (6\underline{i} - 4\underline{j}) + 3(-\underline{i} + 7\underline{j})$

$= -7\underline{i} + 20\underline{j}$

(h) $\frac{1}{3}(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}) = \frac{1}{3}(3\underline{i} - 2\underline{j} + 2\underline{i} - 5\underline{j} + \underline{i} + 2\underline{j})$

$= \frac{1}{3}(6\underline{i} - 5\underline{j}) = 2\underline{i} - \frac{5}{3}\underline{j}$

3/

$\underline{u} = 3\underline{i} + 4\underline{j} - 5\underline{k}$, $\underline{v} = 3\underline{i} - 4\underline{j} - 5\underline{k}$

(a) $\underline{u} + \underline{v} = 6\underline{i} - 10\underline{k}$

$\underline{u} - \underline{v} = 8\underline{j}$

$2\underline{u} - 3\underline{v} = -3\underline{i} + 20\underline{j} + 5\underline{k}$

(b) $|\underline{u}| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{9 + 16 + 25} = \sqrt{50}$

$|\underline{v}| = \sqrt{3^2 + (-4)^2 + (-5)^2} = \sqrt{50}$

(c) $\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|} = \frac{1}{\sqrt{50}}(3\underline{i} + 4\underline{j} - 5\underline{k})$

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{50}}(3\underline{i} - 4\underline{j} - 5\underline{k})$

(d) $\underline{u} \cdot \underline{v} = 3 \cdot 3 + 4 \cdot (-4) + (-5) \cdot (-5) = 9 - 16 + 25$

$= 18.$

(e) $\underset{\sim}{u} \cdot \underset{\sim}{v} = |u||v| \cos\theta \Rightarrow \cos\theta = \dfrac{u \cdot v}{|\underline{u}||\underline{v}|} = \dfrac{18}{50} \Rightarrow \theta \simeq 1.20$
$\simeq 68.9°$

(f) scalar proj. of $\underline{u}$ in direction of $\underline{v}$:

$$S = |\underline{u}| \cos\theta = \sqrt{50} \cdot 18/50 = \frac{18}{\sqrt{50}}$$

(g) $$\underset{\sim}{V}_{\underset{\sim}{u}} = |\underset{\sim}{v}| \cos\theta \cdot \hat{\underset{\sim}{u}} = \sqrt{50} \cdot \frac{18}{50} \cdot \frac{1}{\sqrt{50}} (3\underline{i} + 4j - 5\underline{k})$$

$$= \frac{9}{25} (3\underline{i} + 4\underline{j} - 5\underline{k}).$$

13/ $\underset{\sim}{u} = 2t\underset{\sim}{i} + 4\underset{\sim}{j} - (10+t)\underline{k}, \quad \underset{\sim}{v} = \underset{\sim}{i} + t\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 = 2t + 4t - (10+t) = 5t - 10 \Rightarrow t = 2$

17/ First find a vector that makes equal angles with the three coordaxes.

$\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underline{k}$. $\quad a = \underset{\sim}{u} \cdot \underset{\sim}{i} = |\underline{u}||\underline{i}| \cos\theta_1$

$= |\underline{u}| \cos\theta_1$

$b = \underline{u} \cdot \underset{\sim}{j} = |\underline{u}| \cos\theta_2$

$c = \underline{u} \cdot \underset{\sim}{k} = |\underline{u}| \cos\theta_3$

Now $\theta_1 = \theta_2 = \theta_3 \Rightarrow a = b = c = |\underline{u}| \cos\theta$.

So $\underset{\sim}{u} = a\underset{\sim}{i} + a\underset{\sim}{j} + a\underline{k}$ for some a.

Need $|\underset{\sim}{u}| = 1$, but $|\underline{u}| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}\,a = 1$

$\Rightarrow a = 1/\sqrt{3}$.

$$\underset{\sim}{u} = \frac{1}{\sqrt{3}}\underset{\sim}{i} + \frac{1}{\sqrt{3}}\underset{\sim}{j} + \frac{1}{\sqrt{3}}\underline{k}$$

Note: could also take $\pm\frac{1}{\sqrt{3}}\underset{\sim}{i} \pm \frac{1}{\sqrt{3}}\underset{\sim}{j} \pm \frac{1}{\sqrt{3}}\underline{k}$

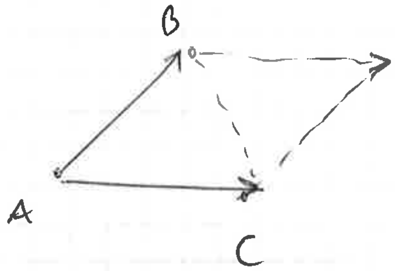
10.3

1/ $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -2 & 3 \\ 3 & 1 & -4 \end{vmatrix} = \underset{\sim}{i}(8-3) - \underline{j}(-4-9) + \underline{k}(1+6)$

$= 5\underset{\sim}{i} + 13\underline{j} + 7\underline{k}$

3/ $A = (1,2,0) \quad B = (1,0,2), \quad C = (0,3,1).$

$\overrightarrow{AB} = (0, -2, 2), \quad \overrightarrow{AC} = (-1, 1, 1)$



$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -2 & 2 \\ -1 & 1 & 1 \end{vmatrix}$

$= \underset{\sim}{i}(-2-2) - \underline{j}(0+2) + \underline{k}(0-2) = -4\underset{\sim}{i} - 2\underline{j} - 2\underline{k}$

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-4)^2 + (-2)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

= area of parallelogram

= 2 × area of triangle

so Area of triangle $= \sqrt{6}$.

5/ $\underset{\sim}{u} = \underline{i} + \underline{j}, \quad \underline{v} = \underline{j} + 2\underline{k}$

$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \underline{i}(2) - \underline{j}(2) + \underline{k}(1)$

$= 2\underline{i} - 2\underline{j} + \underline{k}.$

$|\underline{u} \times \underline{v}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

so $\dfrac{\underline{u} \times \underline{v}}{|\underline{u} \times \underline{v}|} = \frac{1}{3}(2\underline{i} - 2\underline{j} + \underline{k})$ is a unit vector perpendicular to $\underset{\sim}{u}$ and $\underline{v}$.

15/ $A=(1,0,0)$ $B=(1,2,0)$ $C=(2,2,2)$, $D=(0,3,2)$

$\overrightarrow{AB}=(0,2,0)$, $\overrightarrow{AC}=(1,2,2)$, $\overrightarrow{AD}=(-1,3,2)$

$$V = \frac{1}{6}\left|\underset{\sim}{u}\cdot(\underset{\sim}{v}\times\underset{\sim}{w})\right| = \frac{1}{6}\left|\begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 2 \end{vmatrix}\right|$$

$$= \frac{1}{6}\left|0 - 2(2+2) + 0\right| = \frac{1}{6}|-8| = \frac{8}{6} = \frac{4}{3}.$$

21

$$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \underline{i}(3) - \underline{j}(-2) + \underline{k}(2)$$

$$\underline{u}\times(\underline{v}\times\underline{w}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = \underline{i}(-2) - \underline{j}(-7) + \underline{k}(-4)$$

$$\underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 2 & -3 & 0 \end{vmatrix} = \underline{i}(9) - \underline{j}(-6) + \underline{k}(-7)$$

$$(\underline{u}\times\underline{v})\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 9 & 6 & -7 \\ 0 & 1 & -1 \end{vmatrix} = \underline{i}(1) - \underline{j}(-9) + \underline{k}(9)$$

$$\neq \underline{u}\times(\underline{v}\times\underline{w}).$$

In general the cross product is <u>not</u> <u>associative</u>, meaning $\underline{u}\times(\underline{v}\times\underline{w}) \neq (\underline{u}\times\underline{v})\times\underline{w}$.

e.g. $\underset{\sim}{i}\times(\underset{\sim}{i}\times\underline{j}) = \underset{\sim}{i}\times\underline{k} = -\underline{j}$

$(\underset{\sim}{i}\times\underline{i})\times\underline{j} = \underline{0}\times\underline{j} = \underline{0}$.

22/ $\underset{\sim}{u} \cdot \underset{\sim}{v} \times \underset{\sim}{w}$:

We can interpret this in only one way for it to make sense: $\underline{v} \times \underline{w}$ is a vector, then we take dot product with $\underline{u}$. So it equals

$\underset{\sim}{u} \cdot (\underset{\sim}{v} \times \underset{\sim}{w})$.

Note that it does <u>not</u> make sense to take $(\underset{\sim}{u} \cdot \underset{\sim}{v}) \times \underline{w}$ as $\underline{u} \cdot \underset{\sim}{v}$ is a scalar, and we cannot take cross product of a scalar and a vector.

$\underline{u} \times \underline{v} \times \underline{w}$: as we have seen, the two possible interpretations are

$\underset{\sim}{u} \times (\underset{\sim}{v} \times \underset{\sim}{w})$ or $(\underline{u} \times \underline{v}) \times \underline{w}$, but in general these are not equal!

There is no way of knowing which one is meant, so this expression does not make sense (unless we add brackets in the intended position).