10.4 2. Plane through $(0,2,-3)$, normal to $\underset{\sim}{n}=(4,-1,-2)$.
Has equation $(\underset{\sim}{r}-\underset{\sim}{r_0})\cdot\underset{\sim}{n}=0$.

$$(x-0)\cdot 4+(y-2)(-1)+(z-(-3))(-2)=0$$
$$4x-y+2-2z-6=0$$
$$4x-y-2z=4$$

Check: $4(0)-2-2(-3)=-2+6=4$ ✓

4. Plane through $(1,2,3)$, parallel to $3x+y-2z=15$. Both planes have the same normal vector $\underset{\sim}{n}=(3,1,-2)$. Compute the new constant: $3\cdot 1+1\cdot 2+(-2)\cdot 3=-1$.
Equation is $3x+y-2z=-1$.

6. Through points $(-2,0,0),(0,3,0),(0,0,4)$.
Call these points A, B, C. Then $\vec{AB}=(2,3,0)$, $\vec{AC}=(2,0,4)$. Normal vector

$$\underset{\sim}{n}=\vec{AB}\times\vec{AC}=\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 0 \\ 2 & 0 & 4 \end{vmatrix}=\underset{\sim}{i}(12)-\underset{\sim}{j}(8)+\underset{\sim}{k}(-6)$$

Plane has equation $12x-8y-6z=D$.
Substitute any of the points to get D.

$$12(-2)-8(0)-6(0)=D=-24.$$

$$12x-8y-6z=-24$$

Check other two points lie on plane:

$$12(0)-8(3)-6(0)=-24 \checkmark$$
$$12(0)-8(0)-6(4)=-24 \checkmark$$

9. Plane through line defined by the two equations $x+y=2$, $y-z=3$, and perpendicular to the plane $2x+3y+4z=5$.

First find a parametric description of the line.

Let $x=t$, then $y=2-t$, $z=y-3=-1-t$.

Line has parametric description $(x,y,z)=(0,2,-1)+t(1,-1,-1)$.

Passes through $(0,2,-1)$ in direction $(1,-1,-1)$.

Perpendicular to $2x+3y+4z=5$ implies our plane must contain the normal vector $\underset{\sim}{n}=(2,3,4)$.

So the normal vector of our new plane is

$$\underset{\sim}{m}=(1,-1,-1)\times(2,3,4)=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & -1 \\ 2 & 3 & 4 \end{vmatrix} = \underset{\sim}{i}(-1) - \underline{j}\,6 + \underline{k}\,5$$

Equation is $-x-6y+5z=D$. Substitute $(0,2,-1)$.

$$-0-6(2)+5(-1)=D=-17$$

$$\Rightarrow \quad -x-6y+5z=-17.$$

Check: $-t-6(2-t)+5(-1-t)=0t-17$ ✓ $\Rightarrow$ contains line.

$\underset{\sim}{n}\cdot\underset{\sim}{m}=(2,3,4)\cdot(-1,-6,5)=-2-18+20=0$ ✓ planes perpendicular.

15. Line through $(1,2,3)$, parallel to $(2,-3,-4)$.

$\underset{\sim}{r}=(x,y,z)=(1,2,3)+t(2,-3,-4)$, $t\in\mathbb{R}$.

$x=1+2t$, $y=2-3t$, $z=3-4t$ $\quad t\in\mathbb{R}$

$$t=\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{-4}.$$

16. Line through $(-1,0,1)$, perpendicular to $2x-y+7z=12$.

$\underset{\sim}{n} = (2,-1,7)$ is the direction vector of the line.

$$\underset{\sim}{r} = (-1,0,1) + t(2,-1,7) \quad t \in \mathbb{R}$$

$$= -\underset{\sim}{i} + \underset{\sim}{k} + 2t\underset{\sim}{i} - t\underset{\sim}{j} + 7t\underset{\sim}{k}$$

$$x = -1+2t, \quad y = -t, \quad z = 1+7t$$

$$t = \frac{x+1}{2} = \frac{y}{-1} = \frac{z-1}{7}.$$

17. Line through origin, parallel to intersection of

$x+2y-z=2$ and $2x-y+4z=5$.

Find direction vector by crossproduct of normal vectors:

$$\underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 2 & -1 & 4 \end{vmatrix} = \underset{\sim}{i}(7) - \underset{\sim}{j}(6) + \underset{\sim}{k}(-5)$$

$$\underset{\sim}{r} = t(7,-6,-5), \quad t \in \mathbb{R}$$

$$= 7t\underset{\sim}{i} - 6t\underset{\sim}{j} - 5t\underset{\sim}{k} \quad t \in \mathbb{R}$$

$$x = 7t \quad y = -6t \quad z = -5t$$

$$t = x/7 = \frac{y}{-6} = \frac{z}{-5}.$$

<u>12.3</u> 13. Tangent plane & normal line to graph of

$f(x,y) = x^2 - y^2$ at $(-2,1)$. $f(-2,1) = 3$.

$$\partial f/\partial x = 2x, \quad \frac{\partial f}{\partial y} = -2y. \Rightarrow \partial f/\partial x\,(-2,1) = -4, \quad \partial f/\partial y\,(-2,1) = -2.$$

tangent plane: $z = 3 + (-4)(x-(-2)) + (-2)(y-1)$

$$= 3 - 4x - 8 - 2y + 2 = -4x - 2y - 3.$$

normal line: $\underset{\sim}{r} = (-2,1,3) + t(-4,-2,-1), \quad t \in \mathbb{R}.$

14. $f(x,y) = \dfrac{x-y}{x+y}$ at $(1,1)$. $f(1,1) = 0$.

$$\frac{\partial f}{\partial x} = \frac{1(x+y) - (x-y)\cdot 1}{(x+y)^2} = \frac{2y}{(x+y)^2} \Rightarrow \partial f/\partial x\,(1,1) = \frac{2}{4} = \frac{1}{2}.$$

$$\frac{\partial f}{\partial y} = \frac{(-1)(x+y) - (x-y)\cdot 1}{(x+y)^2} = \frac{-2x}{(x+y)^2} \Rightarrow \partial f/\partial y\,(1,1) = \frac{-2}{4} = -\frac{1}{2}.$$

tangent plane: $z = 0 + \frac{1}{2}(x-1) - \frac{1}{2}(y-1)$
$= \frac{1}{2}x - \frac{1}{2}y$

normal line: $\underset{\sim}{r} = (1,1,0) + t\left(\frac{1}{2}, -\frac{1}{2}, -1\right)$, $t \in \mathbb{R}$.

15. $f(x,y) = \cos\left(x/y\right)$ at $(\pi, 4)$. $f(\pi,4) = \cos(\pi/4) = \dfrac{1}{\sqrt{2}}$.

$$\partial f/\partial x = -1/y \sin\left(x/y\right) \Rightarrow \partial f/\partial x\,(\pi,4) = -1/4 \sin(\pi/4) = \frac{-1}{4\sqrt{2}}$$

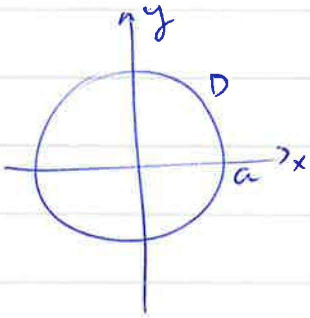
$$\partial f/\partial y = +\frac{x}{y^2}\sin\left(x/y\right) \Rightarrow \partial f/\partial y\,(\pi,4) = \frac{\pi}{16}\sin(\pi/4) = \frac{\pi}{16\sqrt{2}}$$

tangent plane: $z = \dfrac{1}{\sqrt{2}} - \dfrac{1}{4\sqrt{2}}(x-\pi) + \dfrac{\pi}{16\sqrt{2}}(y-4)$.

normal line: $\underset{\sim}{r} = \left(\pi, 4, \frac{1}{\sqrt{2}}\right) + t\left(-\frac{1}{4\sqrt{2}}, \frac{\pi}{16\sqrt{2}}, -1\right)$, $t \in \mathbb{R}$.

$z = \dfrac{1}{\sqrt{2}} - \dfrac{x}{4\sqrt{2}} + \dfrac{\pi y}{16\sqrt{2}}$. tangent plane

14.4 1. $D = \{x^2 + y^2 \le a^2\} = \{[r,\theta] : 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$.



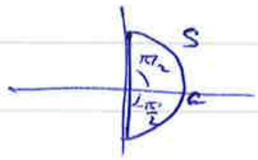
$$\iint_D \underbrace{(x^2+y^2)}_{\substack{r^2\cos^2\theta + r^2\sin^2\theta \\ = r^2}}\, dA = \int_0^{2\pi} d\theta \int_0^a r^2 \cdot r\,dr$$

$$\boxed{r\,dr\,d\theta = dA}$$

$$= \int_0^{2\pi} d\theta\ \tfrac{1}{4} r^4 \Big|_0^a = \tfrac{1}{4} a^4 \cdot 2\pi = \frac{\pi a^4}{2}.$$

2. $$\iint_D \underbrace{\sqrt{x^2+y^2}}_{r}\, dA = \int_0^{2\pi}\int_0^a r \cdot r\,dr\,d\theta = \tfrac{1}{3} a^3 \cdot 2\pi = \frac{2\pi a^3}{3}.$$

4. $I = \iint_D |x|\, dA$. Let $S = \{[r,\theta] : 0 \le r \le a,\ -\pi/2 \le \theta \le \pi/2\}$
= right half of D.

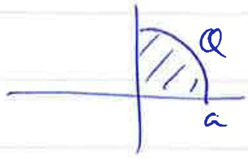


On S, $|x| = x$ as x is positive.
The integral of $|x|$ over the left half of D equals the integral of $|x|$ over the right half S by symmetry.

$$I = 2\iint_S |x|\,dA = 2\int_{-\pi/2}^{\pi/2}\int_0^a r\cos\theta\ r\,dr\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \tfrac{1}{3} r^3 \cos\theta \Big|_0^a d\theta = \tfrac{2}{3} a^3 \sin\theta \Big|_{-\pi/2}^{\pi/2} = \tfrac{2}{3} a^3 \cdot 2 = \frac{4a^3}{3}.$$

8. $Q = \{ x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq a^2 \}$



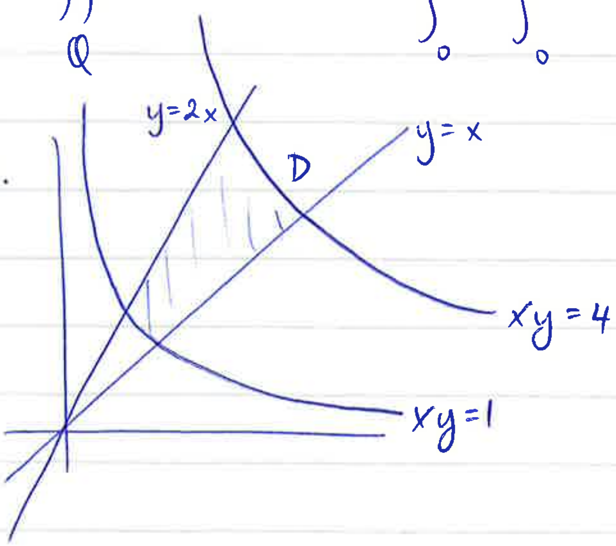
$= \{ [r, \theta] : 0 \leq r \leq a,\ 0 \leq \theta \leq \pi/2 \}.$

$$\iint_Q (x+y)\, dA = \int_0^{\pi/2} \int_0^a r(\cos\theta + \sin\theta) \cdot \underbrace{r\, dr\, d\theta}_{dA}$$

$$= \tfrac{1}{3} a^3 \cdot \left[\sin\theta - \cos\theta\right]_0^{\pi/2} = \tfrac{1}{3} a^3 (1 - 0 - (0 - 1)) = \tfrac{2}{3} a^3$$

9. $$\iint_Q e^{x^2+y^2}\, dA = \int_0^{\pi/2} \int_0^a e^{r^2}\, r\, dr\, d\theta = \tfrac{1}{2}(e^{a^2} - 1) \cdot \frac{\pi}{2}$$

33.



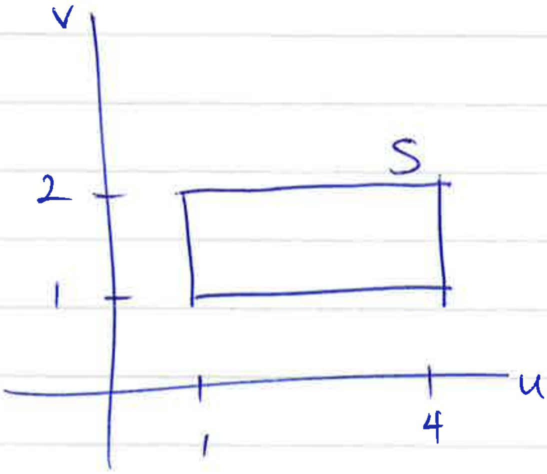
Try substitution $u = xy$, $v = y/x$.

$xy = 1 \Rightarrow u = 1$

$xy = 4 \Rightarrow u = 4$

$y = x \Rightarrow v = y/x = 1$

$y = 2x \Rightarrow v = 2.$



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix}$$

$$= y/x + y/x = 2\, y/x = 2v.$$

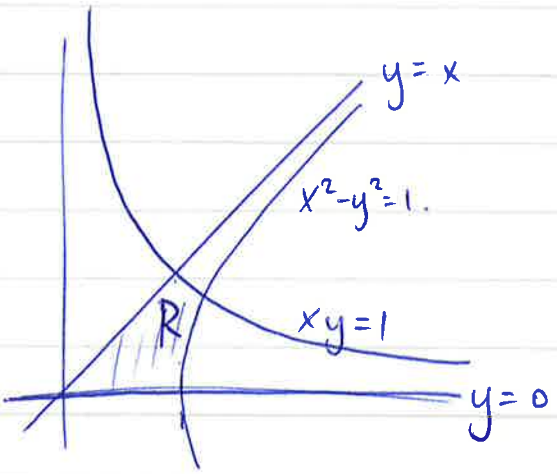
$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v} \Rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{1}{2v} \right| = \frac{1}{2v}$$

since v is already positive on S.

$$A = \iint_D dx\,dy = \int_1^4 du \int_1^2 dv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \int_1^4 du \int_1^2 dv \left(\frac{1}{2v}\right)$$

$$= \frac{1}{2}\int_1^4 du \cdot \log v \Big|_1^2 = \frac{3}{2}(\log 2 - \log 1) = \frac{3}{2}\log 2.$$

34.

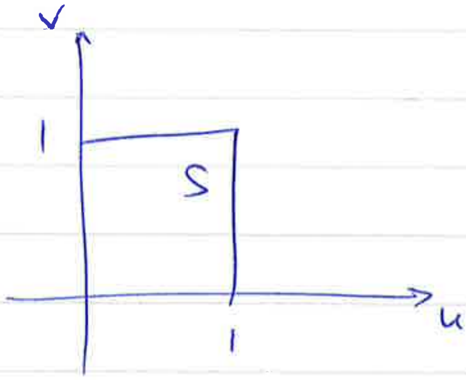


$x^2 - y^2 = 1$

$(x-y)(x+y) = 1$ hyperbola rotated by $\pi/4$ radians.

Put $y=0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$, so it intersects x-axis at $x = \pm 1$.

Try $u = x^2 - y^2$, $v = xy$.



$x^2 - y^2 = 1 \Rightarrow u = 1.$

$y = 0 \Rightarrow v = 0.$

$xy = 1 \Rightarrow v = 1$

$y = x \Rightarrow u = x^2 - y^2 = 0.$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}\right| = |2x^2 + 2y^2| = 2x^2 + 2y^2 \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2(x^2+y^2)}$$ (already positive)

$$\text{So } \iint_R (x^2+y^2)\,dA = \iint_S (x^2+y^2)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{2}\iint_S du\,dv = \frac{1}{2}.$$

area of S in u-v plane.