14.5

5. $$\iiint_R (x^2+y^2)\,dV \qquad 0 \le x, y, z \le 1.$$

$$= \iiint_R x^2\,dV + \iiint_R y^2\,dV$$

note domain is symmetric if we swap any two variables $x \leftrightarrow y$ or $x \leftrightarrow z$ or $y \leftrightarrow z$.

swap $x \leftrightarrow y$ in this integral. R is unchanged due to symmetry, but $y^2 \longrightarrow x^2$.

$$= \iiint_R x^2\,dV + \iiint_R x^2\,dV = 2\iiint_R x^2\,dV$$

$$= 2\int_0^1 dz \int_0^1 dy \int_0^1 x^2\,dx = \tfrac{2}{3}\int_0^1 dz \int_0^1 dy \; x^3\Big|_0^1 = \tfrac{2}{3}$$

6. $$\iiint_R (x^2+y^2+z^2)\,dV$$

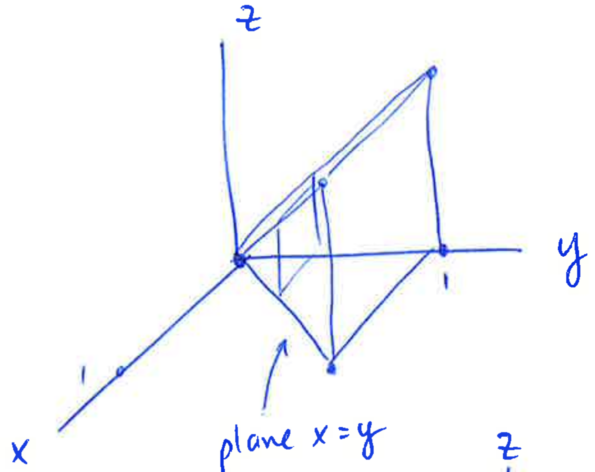
swap $x \leftrightarrow y$

swap $x \leftrightarrow z$

when we swap variables the domain R remains unchanged due to symmetry

$$= 3\iiint_R x^2\,dV = 1$$

9. $\displaystyle\iiint_R \sin(\pi y^3)\, dV$



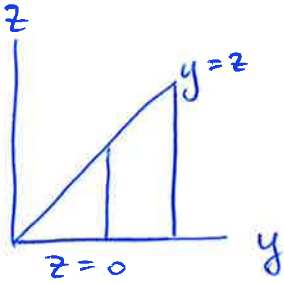
y as outermost variable:

$0 \le y \le 1.$

Next, variable z.

Project pyramid onto yz plane:

Describe limits for z, possibly in terms of y:



$0 \le z \le y$

Now fix y, z, calculate limits for x, possibly in terms of y and z: $0 \le x \le y$.

$$I = \int_0^1 dy \int_0^y dz \int_0^y \sin(\pi y^3)\, dx = \int_0^1 y^2 \sin(\pi y^3)\, dy$$

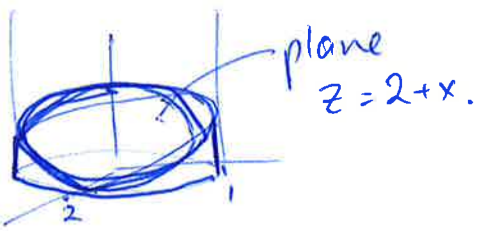
$$= -\frac{1}{3\pi} \cos(\pi y^3)\Big|_0^1 = -\tfrac{1}{3\pi}\left[-1-(1)\right] = \frac{2}{3\pi}$$

14. Inside $x^2 + 4y^2 = 4$, above xy plane $(z \geq 0)$, below plane $z = 2 + x$.

$\left(\frac{x}{2}\right)^2 + y^2 = 1$ so $-2 \le x \le 2$

$-\sqrt{1-\left(\frac{x}{2}\right)^2} \le y \le \sqrt{1-\left(\frac{x}{2}\right)^2}$

$0 \le z \le 2 + x.$



$$V = \int_{-2}^{2} dx \int_{-\sqrt{1-(x/2)^2}}^{\sqrt{1-(x/2)^2}} dy \int_0^{2+x} dz = \underbrace{\int_{-2}^{2} dx \int_{-\sqrt{1-(x/2)^2}}^{\sqrt{1-(x/2)^2}} dy \, (2+x)}$$

this is just $\iint_D (2+x)\, dA$

where D is solid ellipse $x^2 + 4y^2 \leq 4$ in xy plane.

let $u = x/2$, $v = y$, corresponding region S in uv plane is $u^2 + v^2 \leq 1$ disc radius 1.

$$V = \iint_S (2+2u) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \iint_S (2+2u) \cdot 2 \, du\,dv$$

but u is odd function and S is symmetric under reflection $u \mapsto -u$, so

$$V = 4 \iint_S du\,dv = 4 \cdot \text{area}(S) = 4\pi .$$

19. $\int_0^1 dz \int_z^1 dx \int_0^{x-z} f(x,y,z)\, dy$

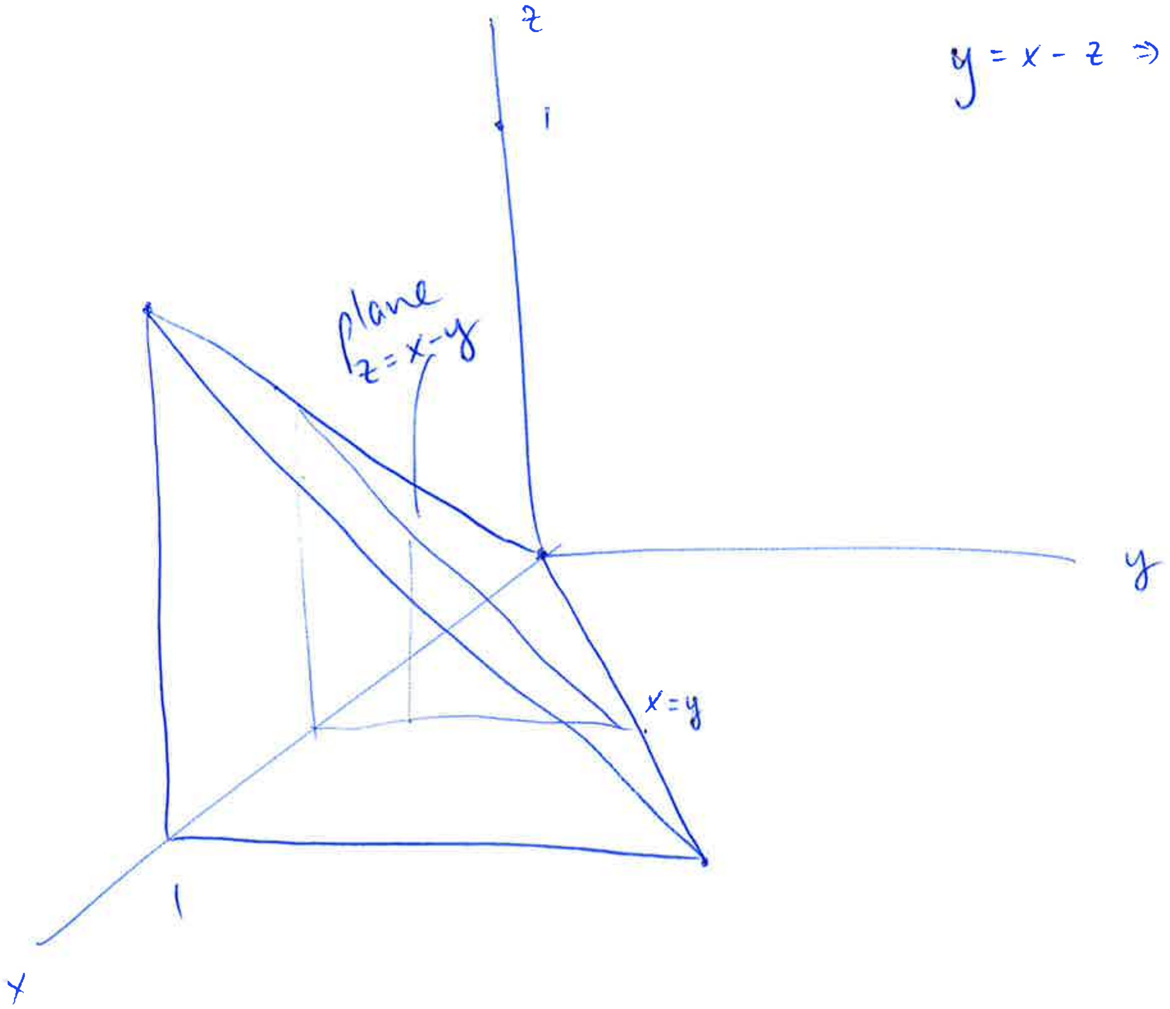
$0 \leq z \leq 1$

$z \leq x \leq 1$

$0 \leq y \leq x - z$

4

$y = x - z \Rightarrow z = x - y$



$0 \le x \le 1$

$0 \le y \le x$

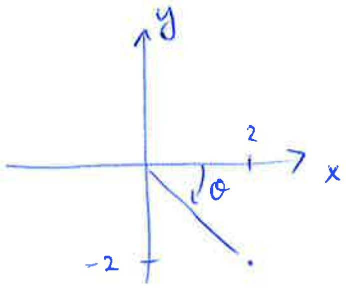
$0 \le z \le x - y$

$$I = \int_0^1 dx \int_0^x dy \int_0^{x-y} f(x,y,z)\, dz$$

10.6

1. $(2,-2,1) = (x,y,z)$

Cylindrical: $r = \sqrt{x^2+y^2} = \sqrt{8} = 2\sqrt{2}$



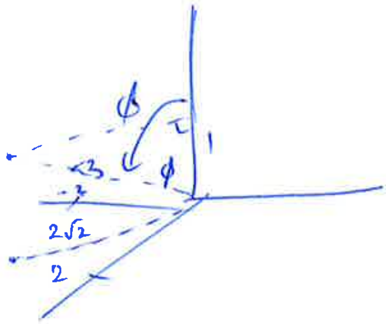
$\theta = -\pi/4$

$z = z = 1$

$[2\sqrt{2}, -\pi/4, 1] = [r, \theta, z]$.

Spherical: $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{9} = 3$.

$\theta = -\pi/4$ as above



$\cos\phi = 1/3$ or $\tan\phi = \frac{2\sqrt{2}}{1}$

$\Rightarrow \phi \approx 1.231$

$[\rho, \phi, \theta] = [3, 1.231, -\pi/4]$

2. $[2, \pi/6, -2] = [r, \theta, z]$ cylindrical.

$x = r\cos\theta = 2\cos\pi/6 = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$y = r\sin\theta = 2\sin\pi/6 = 2 \cdot \frac{1}{2} = 1$

$(x,y,z) = (\sqrt{3}, 1, -2)$.

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{4+4} = 2\sqrt{2}$

$\tan\phi = r/z = \frac{2}{-2} = -1 \Rightarrow \phi = 3\pi/4$

$[\rho, \phi, \theta] = [2\sqrt{2}, \frac{3\pi}{4}, \pi/6]$.

3. $[4, \pi/3, 2\pi/3] = [\rho, \phi, \theta]$

$x = \rho \sin\phi \cos\theta = 4 \sin \pi/3 \cos \frac{2\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right) = -\sqrt{3}$

$y = \rho \sin\phi \sin\theta = 4 \sin \pi/3 \sin 2\pi/3 = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = 3$

$z = \rho \cos\phi = 4 \cos \pi/3 = 4 \cdot \frac{1}{2} = 2$

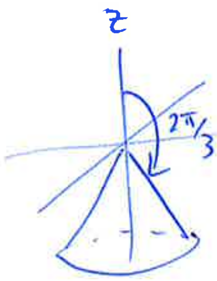
$(x, y, z) = (-\sqrt{3}, 3, 2)$

$r = \sqrt{x^2 + y^2} = \sqrt{3 + 9} = 2\sqrt{3}$

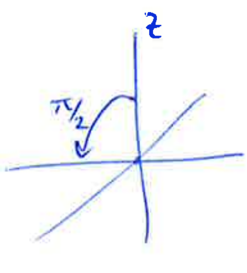
$[r, \theta, z] = [2\sqrt{3}, 2\pi/3, 2]$.

5. $\theta = \frac{\pi}{2}$ half plane, vertical, angle $\theta = \pi/2$ to x axis, edge along z-axis.

6. $\phi = \frac{2\pi}{3}$ half-cone, point at origin, lines from origin at angle $2\pi/3$ to positive z-axis



7. $\phi = \pi/2$



the entire ~~xy~~ plane (all points whose position vector is at angle $\pi/2$ to z-axis)

8. $R = 4$ $(\rho = 4)$ sphere radius 4 centred at origin

9. $r = 4$ cylinder along z-axis, radius 4.

10. $R = z$ $(\rho = z)$ $\Rightarrow \sqrt{x^2+y^2+z^2} = z \Rightarrow x^2+y^2+z^2 = z^2$
$\Rightarrow x^2 + y^2 = 0 \Rightarrow x = y = 0 \Rightarrow$ positive z-axis.

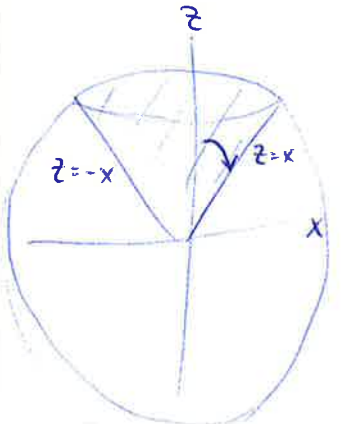
11. $R = r \Rightarrow \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2} \Rightarrow x^2+y^2+z^2 = x^2+y^2$

$\Rightarrow z^2 = 0 \Rightarrow z = 0 \Rightarrow$ xy plane.

14.6

1. Volume inside cone $z = \sqrt{x^2+y^2} = r$

and inside sphere $x^2+y^2+z^2 = a^2$.



Use spherical coords: $[R, \phi, \theta]$

$0 \leq R \leq a$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{4}$

$$\boxed{dV = R^2 \sin\phi \, dR\, d\phi\, d\theta}$$

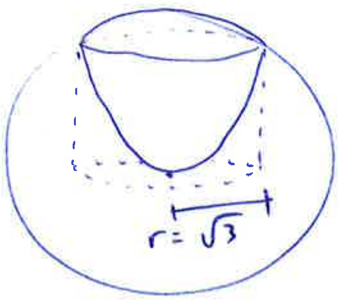
$$V = \int_0^a dR \int_0^{2\pi} d\theta \int_0^{\pi/4} R^2 \sin\phi \, d\phi$$

$$= \int_0^a R^2 dR \int_0^{2\pi} d\theta \cdot \left(-\cos\phi\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{3} a^3 \cdot 2\pi \cdot \left[1 - \frac{1}{\sqrt{2}}\right]$$

4. Inside $z = x^2+y^2$ and ~~insider~~ inside $x^2+y^2+z^2 = 12$

Try cylindrical coords.



Intersection: $z + z^2 = 12$

$z^2 + z - 12 = 0$

$(z+4)(z-3)$

$z = -4$ or $z = 3$ but $z = x^2+y^2$ is positive

$\Rightarrow z = 3$.

$\Rightarrow x^2+y^2 = r^2 = 3 \Rightarrow r = \sqrt{3}$.

$0 \leq r \leq \sqrt{3}$

$0 \leq \theta \leq 2\pi$

$x^2+y^2 \leq z \leq \sqrt{12 - x^2+y^2}$ i.e. $r^2 \leq z \leq \sqrt{12-r^2}$

$\boxed{dV = r\,dr\,d\theta\,dz}$

$$V = \int_0^{\sqrt{3}} dr \int_0^{2\pi} d\theta \int_{r^2}^{\sqrt{12-r^2}} r\,dz$$

$$= \int_0^{\sqrt{3}} dr \cdot 2\pi \cdot r\left(\sqrt{12-r^2} - r^2\right)$$

$$= 2\pi \int_0^{\sqrt{3}} r\sqrt{12-r^2} - r^3\,dr = 2\pi\left[\tfrac{2}{3}\left(-\tfrac{1}{2}\right)\left(12-r^2\right)^{3/2} - \tfrac{1}{4}r^4\right]_0^{\sqrt{3}}$$

$$= 2\pi\left[-\tfrac{1}{3}(9)^{3/2} - \tfrac{1}{4}\cdot 9 + \tfrac{1}{3}(12)^{3/2}\right]$$

$$= 2\pi\left[-9 - \tfrac{9}{4} + 8\sqrt{3}\right] \quad (>0)$$

11. $\iiint_B (x^2+y^2)\,dV$ $\quad B: \; x^2+y^2+z^2 \leq a^2.$

Spherical : $0 \leq R \leq a$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

By symmetry under exchange of variables $x \leftrightarrow y \leftrightarrow z$

$$I = 2\iiint_B \underbrace{z^2}_{(R\cos\phi)^2}\,dV = 2\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^a R^2\cos^2\phi \cdot R^2 \sin\phi\,dR.$$

$$= 4\pi \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \int_0^{a} R^4 \, dR$$

$$= 4\pi \left(-\frac{1}{3}\cos^3\phi\right)\Big|_0^{\pi} \cdot \frac{1}{5} a^5$$

$$= 4 \frac{\pi a^5}{15}(1+1) = \frac{8\pi a^5}{15}$$

15. $\iiint_R z \, dV$ $\quad R: \ x^2+y^2 \leq z \leq \sqrt{2-x^2-y^2}$

try cylindrical : $r^2 \leq z \leq \sqrt{2-r^2}$

$0 \leq \theta \leq 2\pi$

intersection $r^2 = \sqrt{2-r^2} \Rightarrow r^4 = 2-r^2 \Rightarrow r^4+r^2-2=0$

$(r^2+2)(r^2-1)=0$

$\Rightarrow r=1$

$\Rightarrow 0 \leq r \leq 1.$

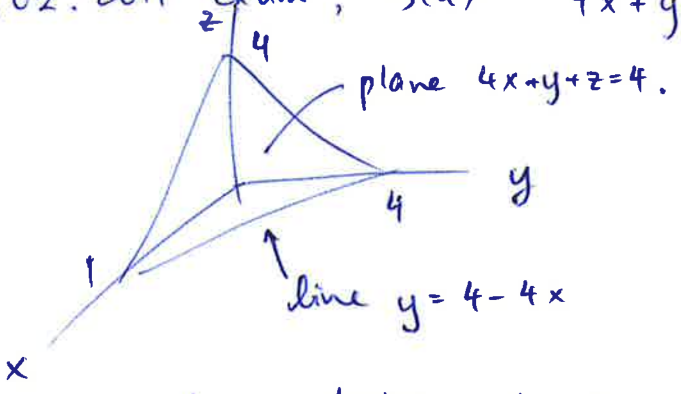
$$\int_0^{2\pi} d\theta \int_0^1 dr \int_{r^2}^{\sqrt{2-r^2}} z \, r \, dz$$

$$= \frac{2\pi}{2} \int_0^1 dr \ r z^2 \Big|_{r^2}^{\sqrt{2-r^2}} = \pi \int_0^1 dr \left(r(2-r^2) - r^3 \right)$$

$$= \pi \left[\frac{1}{2}\left(-\frac{1}{2}\right)(2-r^2)^2 - \frac{1}{4} r^4 \right]_0^1$$

$$= \pi \left[-\frac{1}{4} - \frac{1}{4} + \frac{1}{4} \cdot 4 \right] = \frac{\pi}{2}.$$

28.02.2011 exam, 3(a) $4x+y+z=4$, $x,y,z \geq 0$ (1. octant) ⑩



$0 \leq x \leq 1$

$0 \leq y \leq 4-4x$

$0 \leq z \leq 4-y-4x$.

$$\int_0^1 dx \int_0^{4-4x} dy \int_0^{4-y-4x} x \, dz$$

$$= \int_0^1 dx \int_0^{4-4x} dy \left(4x - xy - 4x^2\right)$$

$$= \int_0^1 dx \left(4xy - \tfrac{1}{2}xy^2 - 4x^2 y\right)\Big|_0^{4-4x}$$

$$= \int_0^1 dx \left(16x - 16x^2 - \tfrac{1}{2}x(16 - 32x + 16x^2) - 4x^2(4-4x)\right)$$

$$= 8x^2 - \tfrac{16}{3}x^3 - 4x^2 + \tfrac{16}{3}x^3 - 2x^4 - \tfrac{16}{3}x^3 + 4x^4 \Big|_0^1$$

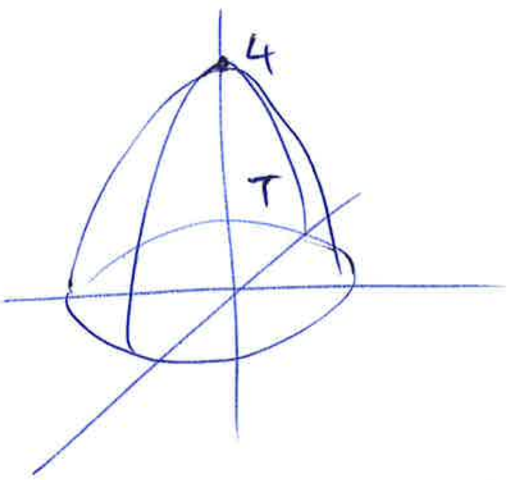
$$= 8 - \cancel{\tfrac{16}{3}} - \cancel{4} + \cancel{\tfrac{16}{3}} - 2 - \tfrac{16}{3} + \cancel{4} = 6 - \tfrac{16}{3} = \tfrac{2}{3}.$$

12.12.2013 4(a) $z = 4 - x^2 - y^2$, $z \geqslant 0$.

T is region under $z = 4 - x^2 - y^2$ and above xy plane.

(a) $z = 0 \Rightarrow x^2 + y^2 = 4$ circle radius 2 centre at origin.

Easiest to use cylindrical coords due to axial symmetry.



$0 \leqslant r \leqslant 2$

$0 \leqslant \theta \leqslant 2\pi$

$0 \leqslant z \leqslant 4 - r^2$ $\boxed{dV = r\,dr\,d\theta\,dz}$

$$\int_0^{2\pi} d\theta \int_0^2 dr \int_0^{4-r^2} z^2 r\, dz$$

$$= \frac{2\pi}{3} \int_0^2 dr \left. z^3 r \right|_0^{4-r^2}$$

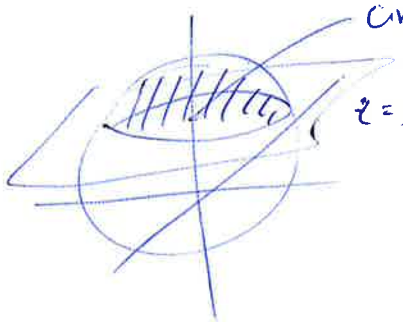
$$= \frac{2\pi}{3} \int_0^2 (4-r^2)^3 r\, dr$$

$$= \frac{2\pi}{3} \cdot \frac{1}{4} \cdot \left(-\frac{1}{2}\right) \left[(4-r^2)^4 \right] \Big|_0^2$$

$$= -\frac{\pi}{12} \left[0 - 256 \right] = \frac{64}{3}\pi$$

2015. 03.03. 4(a). between $x^2+y^2+z^2=25$ and $z=3$ plane.

$z=3 \Rightarrow \quad x^2+y^2=16$ circle radius 4.



Use cylindrical coords.

$0 \le \theta \le 2\pi$

$0 \le r \le 4$

$3 \le z \le \sqrt{25-r^2}$

$$\int_0^{2\pi} d\theta \int_0^4 dr \int_3^{\sqrt{25-r^2}} z \cdot r\,dz \qquad \boxed{dV = r\,dr\,d\theta\,dz}$$

$$= \frac{2\pi}{2} \int_0^4 dr\,(25-r^2-9)\,r = \pi \int_0^4 (16r - r^3)\,dr$$

$$= \pi\left[8r^2 - \tfrac{1}{4}r^4\right]_0^4 = \pi\left[8\cdot 16 - 64\right] = 64\pi$$