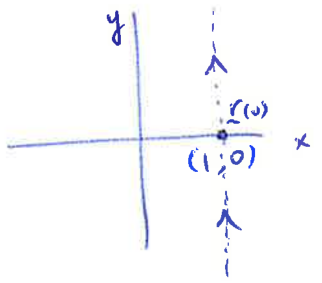
11.1

1. $\underset{\sim}{r} = \underset{\sim}{i} + t\underset{\sim}{j}$, $\quad \underset{\sim}{v} = \underset{\sim}{r}' = \underset{\sim}{j}$, $\underset{\sim}{a} = \underset{\sim}{r}'' = \underset{\sim}{0}$

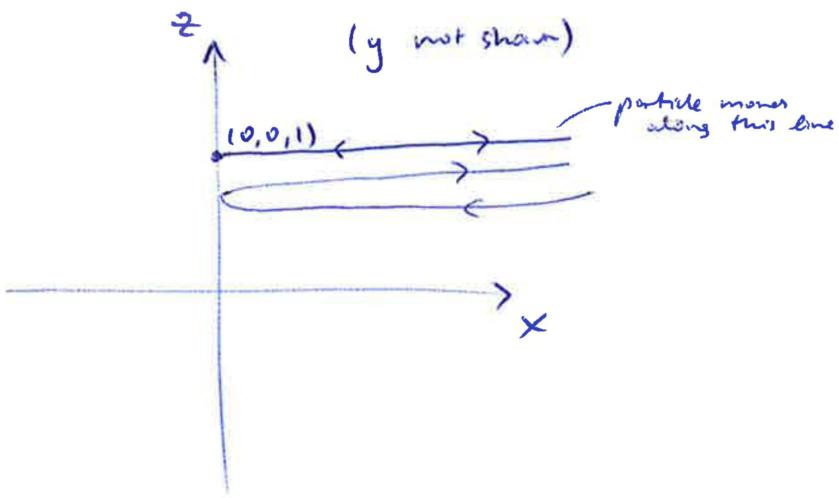
$v = |\underset{\sim}{v}| = 1$

$= (1,0) + t(0,1)$ straight line through $(1,0)$ in direction $(0,1)$. Moving in positive y at constant speed 1.



2. $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + \underset{\sim}{k}$, $\quad \underset{\sim}{v} = \underset{\sim}{r}'(t) = 2t\underset{\sim}{i}$ $\quad v = |\underset{\sim}{v}| = 2|t|$, $\underset{\sim}{a} = \underset{\sim}{r}'' = 2\underset{\sim}{i}$.

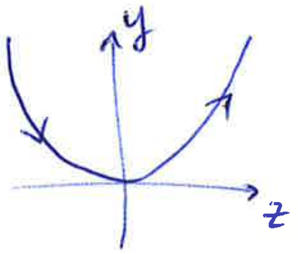
straight line starting at $(0,0,1) = \underset{\sim}{k}$ $(t=0)$ going in positive x-direction only. As t is negative and increasing towards zero, the particle comes in along this line from infinity. At $t=0$ the particle is stationary $(v=0)$ at $(0,0,1)$. As t increases the particle accelerates back out along the same line.



3. $\underline{r} = t^2 \underline{j} + t\underline{k}$ $\underline{r}' = \underline{v} = 2t\underline{j} + \underline{k}$ $v = |\underline{v}| = \sqrt{4t^2+1}$ $\underline{a} = 2\underline{j}$

let $y(t) = t^2$, $z(t) = t$ then motion satisfies $y = z^2 \Rightarrow$ parabola.

At $t=0$ particle at origin.



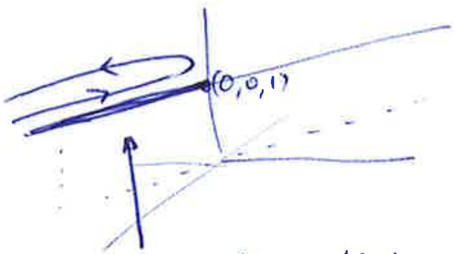
4. $\underline{r} = \underline{i} + t\underline{j} + t\underline{k}$ $\underline{v} = \underline{r}' = \underline{j} + \underline{k}$ $\underline{a} = \underline{0}$ $v = \sqrt{2}$

$= (1,0,0) + t(0,1,1)$ particle moving with constant velocity along line through $(1,0,0)$ in direction $(0,1,1)$.

5. $\underline{r} = t^2\underline{i} - t^2\underline{j} + \underline{k}$ $\underline{v} = \underline{r}' = 2t\underline{i} - 2t\underline{j}$ $\underline{a} = 2\underline{i} - 2\underline{j}$ $v = \sqrt{8t^2} = \sqrt{8}|t|$.

Moves along ray from $(0,0,1)$ in direction $(1,-1,0)$.

Comes in from infinity, stops at $(0,0,1)$ then heads out to infinity along the same ray.



moves along this ray.

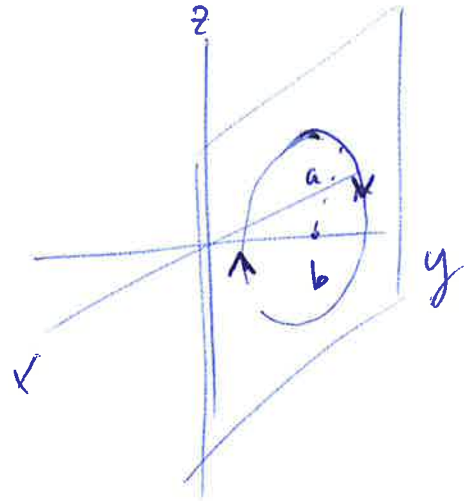
8. $\underset{\sim}{r} = a\cos\omega t\, \underset{\sim}{i} + b\, \underset{\sim}{j} + a\sin\omega t\, \underset{\sim}{k}$ (assume $a>0$ $\omega>0$)

$\underset{\sim}{v} = \underset{\sim}{r}' = -a\omega\sin\omega t\, \underset{\sim}{i} + a\omega\cos\omega t\, \underset{\sim}{k}$ $v = |\underset{\sim}{v}| = a\omega$

$\underset{\sim}{a} = \underset{\sim}{r}'' = -a\omega^2\cos\omega t\, \underset{\sim}{i} - a\omega^2\sin\omega t\, \underset{\sim}{k} = -\omega^2(\underset{\sim}{r} - b\underset{\sim}{j})$

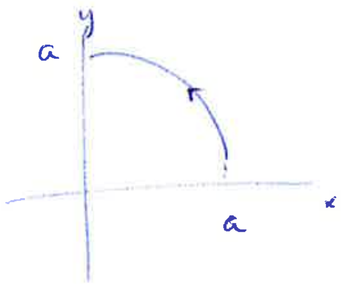
Circular motion in $y=b$ plane about point $(0,b,0)$

radius a, angular velocity ω.

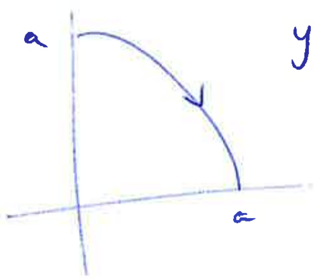


11.3 1. $x^2 + y^2 = a^2$ $\underset{\sim}{r}(y) = \sqrt{a^2-y^2}\, \underset{\sim}{i} + y\underset{\sim}{j}$, $y \in [0,a]$.

$x = \sqrt{a^2-y^2}$



2.



$y = \sqrt{a^2-x^2}$ $\underset{\sim}{r}(x) = x\underset{\sim}{i} + \sqrt{a^2-x^2}\,\underset{\sim}{j}$, $x \in [0,a]$

5. $z = x^2 \quad z = 4y^2 \Rightarrow x^2 = 4y^2 \Rightarrow x^2 - 4y^2 = 0$

$\Rightarrow (x-2y)(x+2y) = 0. \Rightarrow x = 2y$ or $x = -2y.$

At point $(2,-1,4)$, $x = -2y$ passes through this point.

Let $y = t \Rightarrow x = -2t \qquad \underset{\sim}{r}(t) = -2t\,\underset{\sim}{i} + t\,\underset{\sim}{j}, \quad t \in \mathbb{R}.$

7. $x^2 + y^2 = 9 \qquad z = x + y.$ Parameterise cylinder first:

$x = 3\cos t, \quad y = 3\sin t \Rightarrow x^2 + y^2 = 9.$

$z = 3\cos t + 3\sin t \Rightarrow \underset{\sim}{r}(t) = 3\cos t\,\underset{\sim}{i} + 3\sin t\,\underset{\sim}{j} + (3\cos t + 3\sin t)\underset{\sim}{k}$

13. $\underset{\sim}{r} = t^2\underset{\sim}{i} + t^2\underset{\sim}{j} + t^3\underset{\sim}{k} \Rightarrow \underset{\sim}{v} = \underset{\sim}{r}' = 2t\underset{\sim}{i} + 2t\underset{\sim}{j} + 3t^2\underset{\sim}{k}$

$v = |\underset{\sim}{v}| = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = \sqrt{8t^2 + 9t^4} = t\sqrt{8 + 9t^2}$

$$S = \int_0^1 v\,dt = \int_0^1 t\,(8 + 9t^2)^{1/2}\,dt = \frac{2}{3}\cdot\frac{1}{18}(8 + 9t^2)^{3/2}\Big|_0^1$$

$$= \frac{1}{27}\left[17^{3/2} - 8^{3/2}\right].$$

15. $\underset{\sim}{r} = at^2\underset{\sim}{i} + bt\underset{\sim}{j} + c\ln t\,\underset{\sim}{k} \qquad 1 \le t \le T$

$\underset{\sim}{v} = 2at\,\underset{\sim}{i} + b\underset{\sim}{j} + \frac{c}{t}\underset{\sim}{k}$

$v = \sqrt{4a^2t^2 + b^2 + c^2/t^2}$

$$S = \int_1^T \frac{1}{t}\sqrt{4a^2t^4 + b^2t^2 + c^2}\,dt$$

$b^2 = 4ac$ $\qquad 4a^2t^4 + 4act^2 + c^2 = (2at^2+c)^2$

$$S = \int_1^T \tfrac{1}{t}(2at^2+c)\,dt = \int_1^T 2at + \tfrac{c}{t}\,dt$$

$$= \left[at^2 + c\ln t\right]\Big|_1^T = aT^2 - a + c\ln T$$

<u>15.3</u> 4. $C: \quad x = t\cos t, \quad y = t\sin t, \quad z = t. \qquad 0 \le t \le 2\pi$

Find $\int_C z\,ds$. $\qquad \underset{\sim}{r}(t) = t\cos t\,\underset{\sim}{i} + t\sin t\,\underset{\sim}{j} + t\,\underset{\sim}{k}$

$$\underset{\sim}{v} = \underset{\sim}{r}'(t) = (\cos t - t\sin t)\underset{\sim}{i} + (\sin t + t\cos t)\underset{\sim}{j} + \underset{\sim}{k}$$

$$v = |\underset{\sim}{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}$$

$$= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1}$$

$$= \sqrt{1 + t^2 + 1} = \sqrt{2+t^2}$$

$$\int_C z\underbrace{ds}_{\substack{ds = v\,dt \\ = \left|\frac{dr}{dt}\right|dt}} = \int_0^{2\pi} \overset{z(t)}{t} \cdot \underbrace{\sqrt{2+t^2}}_{\left|\frac{dr}{dt}\right|}\,dt = \frac{2}{3}\cdot\frac{1}{2}(2+t^2)^{3/2}\Big|_0^{2\pi}$$

$$= \tfrac{1}{3}\left[(2+4\pi^2)^{3/2} - 2^{3/2}\right]$$

5. $\underset{\sim}{r} = 3t\,\underset{\sim}{i} + 3t^2\,\underset{\sim}{j} + 2t^3\,\underset{\sim}{k}$ $\quad 0 \le t \le 1$

density $\delta(t) = 1 + t$.

$\underset{\sim}{v} = 3\underset{\sim}{i} + 6t\,\underset{\sim}{j} + 6t^2\underset{\sim}{k}$ $\quad v = |\underset{\sim}{v}| = \sqrt{9 + 36t^2 + 36t^4}$

$= \sqrt{(6t^2+3)^2}$

$= 6t^2 + 3$

$$m = \int_C \delta\, ds = \int_0^1 (1+t)\cdot \overbrace{(6t^2+3)}^{v}\, dt = \int_0^1 6t^2 + 3 + 6t^3 + 3t \; dt$$

$$= 2t^3 + 3t + \frac{3}{2}t^4 + \frac{3}{2}t^2 \Big|_0^1 = 2 + 3 + \frac{3}{2} + \frac{3}{2} = 11g$$

9. $x - y + z = 0$, $\quad x + y + 2z = 0$. $\quad (0,0,0)$ to $(3,1,-2)$

$\Rightarrow$ $2x + 3z = 0$

Let $y = t$ $\Rightarrow$ $2x - 2y + 2z = 0 \Rightarrow -3z - 2y + 2z = 0$

$\Rightarrow z = -2y = -2t$

$\Rightarrow x = -\frac{3}{2}z = 3t$.

$\underset{\sim}{r}(t) = 3t\underset{\sim}{i} + t\underset{\sim}{j} - 2t\underset{\sim}{k}$, $\quad t \in [0,1]$. $\Rightarrow \underset{\sim}{v}(t) = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$v = \sqrt{9+1+4} = \sqrt{14}$

$$\int_C x^2\, ds = \int_0^1 9t^2\sqrt{14}\, dt = 3\sqrt{14}\; t^3\Big|_0^1 = 3\sqrt{14}.$$

11. $x = \cos t \quad y = \sin t \quad z = t \quad t \in [0, 2\pi]$.

$\delta(x,y,z) = z$.

$\underset{\sim}{r}(t) = \cos t \, \underset{\sim}{i} + \sin t \, \underset{\sim}{j} + t \, \underset{\sim}{k}$

$\underset{\sim}{v}(t) = -\sin t \, \underset{\sim}{i} + \cos t \, \underset{\sim}{j} + \underset{\sim}{k} \qquad v = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$m = \int_0^{2\pi} \overset{\delta(x,y,z)}{z} \cdot \overset{v}{\sqrt{2}} \, dt = \sqrt{2}\int_0^{2\pi} t \, dt = \frac{1}{\sqrt{2}} t^2 \Big|_0^{2\pi} = 2\sqrt{2}\pi^2$$

$\boxed{p. 851}$

$$M_{x=0} = \int_0^{2\pi} x \, z\sqrt{2} \, dt = \sqrt{2}\int_0^{2\pi} t \cos t \, dt = {}^{+}\sqrt{2}\left[t\sin t + \cos t\right]_0^{2\pi}$$

$u = t \qquad v = +\sin t$

$du = dt \qquad dv = +\cos t \, dt$

$$\int t\cos t \, dt = \int u \, dv = uv - \int v \, du = +t\sin t + \int \sin t \, dt = +t\sin t + \cos t$$

$$M_{x=0} = -\sqrt{2}[1-1] = 0 \Rightarrow \bar{x} = \frac{M_{x=0}}{m} = 0$$

$$M_{y=0} = \int_0^{2\pi} y \, z\sqrt{2} \, dt = \int_0 t\sin t \sqrt{2} \, dt = \sqrt{2}(-t\cos t + \sin t)_0^{2\pi}$$

$u = t \qquad v = -\cos t$

$du = dt \qquad dv = \sin t \, dt$

$$\int u \, dv = uv - \int v \, du = -t\cos t + \int \cos t \, dt = -t\cos t + \sin t$$

$$M_{y=0} = \sqrt{2}(-2\pi - (0)) = -2\sqrt{2}\pi. \qquad \bar{y} = \frac{M_{y=0}}{m} = -\frac{1}{\pi}$$

$$M_{z=0} = \int_0^{2\pi} z \cdot z \cdot \sqrt{2} \, dt = \int_0^{2\pi} \sqrt{2} t^2 \, dt = \frac{\sqrt{2}}{3} t^3 \Big|_0^{2\pi} = \frac{\sqrt{2}}{3} 8\pi^3$$

$$\bar{z} = \frac{M_{z=0}}{m} = \frac{4}{3}\pi$$

12.7 1. $f(x,y) = x^2 - y^2$ at $(2,-1)$. $f(2,-1) = 4 - 1 = 3$.

$\nabla f(x,y) = 2x\,\mathbf{i} - 2y\,\mathbf{j}$ $\qquad \nabla f(2,-1) = 4\,\mathbf{i} + 2\,\mathbf{j}$

Recall tangent plane has normal vector

$$\mathbf{n} = \frac{\partial f}{\partial x}(2,-1)\,\mathbf{i} + \frac{\partial f}{\partial y}(2,-1)\,\mathbf{j} - \mathbf{k} = \nabla f(2,-1) - \mathbf{k}$$

$$= 4\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$$

plane has equation $4x + 2y - z = 4(2) + 2(-1) - 3$
$= 3$

level curve is $f(x,y) = f(2,-1) = 3 \Rightarrow x^2 - y^2 = 3$.

$\nabla f(2,-1)$ is normal to this level curve at $(2,-1)$

$\Rightarrow$ ~~tangent~~ line has equation $(x-2)\cdot 4 + (y+1)\cdot 2 = 0$

$4x + 2y = 6$ or $2x + y = 3$

7. $f(x,y,z) = x^2 y + y^2 z + z^2 x$ $\qquad (1,-1,1)$

$\nabla f = (2xy + z^2)\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + (y^2 + 2zx)\,\mathbf{k}$

$\nabla f(1,-1,1) = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ normal vector to level surface of f through $(1,-1,1)$.

$\Rightarrow$ tangent plane is $-x - y + 3z = -1 - (-1) + 3(1)$
$= 3$.

9. $f(x,y,z) = y e^{-x^2} \sin z \qquad (0, 1, \pi/3)$

$\nabla f(x,y,z) = -2xy e^{-x^2} \sin z \, \mathbf{i} + e^{-x^2} \sin z \, \mathbf{j} + y e^{-x^2} \cos z \, \mathbf{k}$

$\nabla f(0, 1, \pi/3) = \frac{\sqrt{3}}{2} \mathbf{j} + \frac{1}{2} \mathbf{k}$

tangent plane $\quad \frac{\sqrt{3}}{2} y + \frac{1}{2} z = \frac{\sqrt{3}}{2}(1) + \frac{1}{2}(\pi/3) = \frac{\sqrt{3}}{2} + \pi/6.$

10. $f(x,y) = 3x - 4y$ at $(0,2)$ in direction $-2\mathbf{i}$

$\nabla f = 3\mathbf{i} - 4\mathbf{j} \qquad$ unit vector $\hat{\mathbf{u}} = -\mathbf{i}$

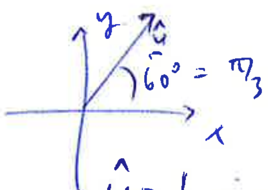
$D_{-2\mathbf{i}} f = \hat{\mathbf{u}} \cdot \nabla f(0,2) = -\mathbf{i} \cdot (3\mathbf{i} - 4\mathbf{j}) = -3 \quad \left(= -\frac{\partial f}{\partial x}(0,2) \right)$

11. $f(x,y) = x^2 y$ at $(-1,-1)$ in direction $\mathbf{i} + 2\mathbf{j}$

$\nabla f(x,y) = 2xy\,\mathbf{i} + x^2 \mathbf{j} \qquad \nabla f(-1,-1) = 2\mathbf{i} + \mathbf{j}$

$\hat{\mathbf{u}} = \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} \qquad D_{\hat{\mathbf{u}}} f = \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} \cdot (2\mathbf{i} + \mathbf{j}) = \frac{4}{\sqrt{5}}$

13. $f(x,y) = x^2 + y^2$ at $(1,-2)$



$\hat{\mathbf{u}} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

$\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j}$

$\nabla f(1,-2) = 2\mathbf{i} - 4\mathbf{j}$

$D_{\hat{\mathbf{u}}} f = \left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) \cdot (2\mathbf{i} - 4\mathbf{j}) = 1 - 2\sqrt{3}$