12.7 ex17. $f(x,y) = xy \Rightarrow \underset{\sim}{\nabla} f(x,y) = y\underset{\sim}{i} + x\underset{\sim}{j}$

$\underset{\sim}{\nabla} f(2,0) = 2\underset{\sim}{j}$

want $\underset{\sim}{v}$ s.t. $D_{\underset{\sim}{v}} f = \underset{\sim}{\nabla} f \cdot \underset{\sim}{v} = -1$, $|\underset{\sim}{v}| = 1$.

Let $\underset{\sim}{v} = \cos\theta\, \underset{\sim}{i} + \sin\theta\, \underset{\sim}{j} \Rightarrow |\underset{\sim}{v}| = 1$. Then

$$\underset{\sim}{\nabla} f \cdot \underset{\sim}{v} = 2\sin\theta = -1 \Rightarrow \sin\theta = -\tfrac{1}{2}$$
$$\Rightarrow \cos\theta = \pm \tfrac{\sqrt{3}}{2}$$

So $\underset{\sim}{v} = \frac{\sqrt{3}}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$ or $\underset{\sim}{v} = -\frac{\sqrt{3}}{2} - \frac{1}{2}\underset{\sim}{j}$.

want $D_{\underset{\sim}{w}} f = -3 = \underset{\sim}{\nabla} f \cdot \underset{\sim}{w}$. $-|\underset{\sim}{\nabla} f| \le D_{\underset{\sim}{w}} f \le |\underset{\sim}{\nabla} f|$

$-2 \le D_{\underset{\sim}{w}} f \le 2$

So no such direction $\underset{\sim}{w}$ exists.

want $D_{\underset{\sim}{w}} f = -2 = \underset{\sim}{\nabla} f \cdot \underset{\sim}{w} \Rightarrow \underset{\sim}{w} = -\underset{\sim}{j}$, unique direction in which f decreases at maximum rate possible.

Ch. 12 review, 6(a) $S: z = f(x,y)$, $f(x,y) = e^{x^2 - 2x - 4y^2 + 5}$

$f(1,-1) = e^0 = 1$ ✓

Let $F(x,y,z) = e^{x^2-2x-4y^2+5} - z$, then

$S: F(x,y,z) = 0$.

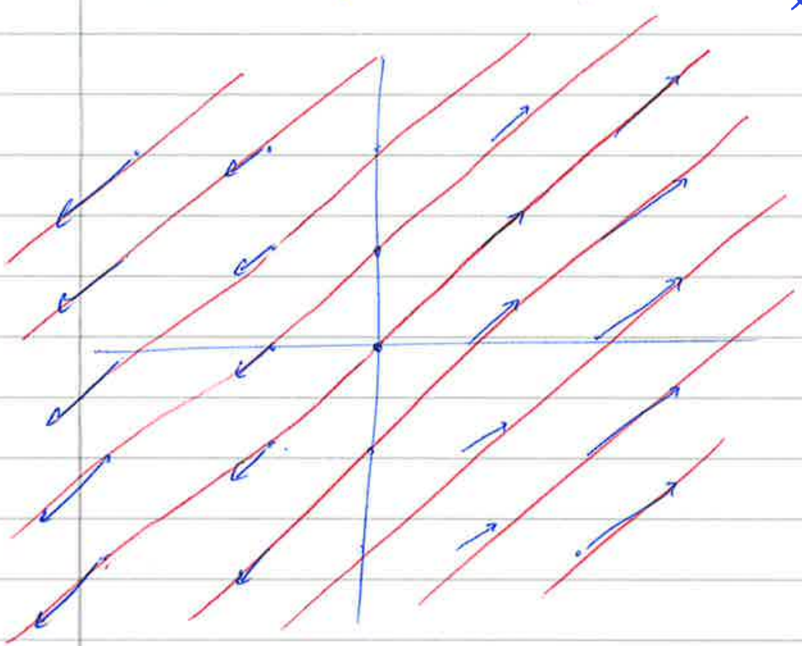
$$\underset{\sim}{\nabla} F = (2x-2)e^{x^2-2x+4y^2+5}\underset{\sim}{i} - 8y\, e^{x^2-2x-4y^2+5}\underset{\sim}{j} - \underset{\sim}{k}$$

$\underset{\sim}{\nabla} F(1,-1,1) = 0\underset{\sim}{i} + 8\underset{\sim}{j} - \underset{\sim}{k}$ normal vector to tangent plane T.

$T: \quad 8y - z = 8(-1) - (1) = -9.$

15.1 (1) $\underset{\sim}{F}(x,y) = x\underset{\sim}{i} + x\underset{\sim}{j}$ $\frac{dx}{x} = \frac{dy}{x} \Rightarrow dx = dy$

$\Rightarrow x = y + c$

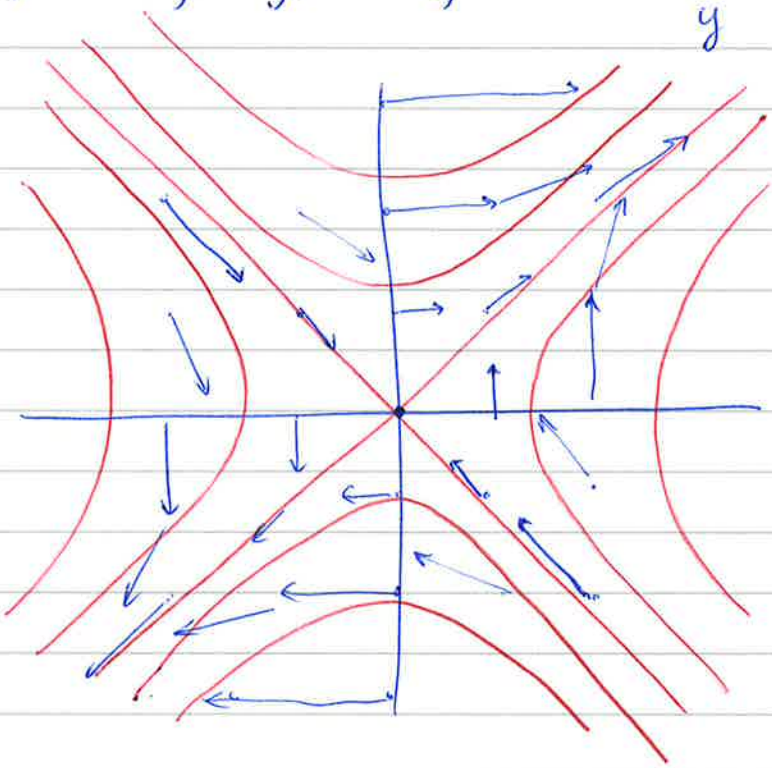


(3) $F(x,y) = y\underset{\sim}{i} + x\underset{\sim}{j}$ $\frac{dx}{y} = \frac{dy}{x} \Rightarrow xdx = ydy$

$\Rightarrow x^2 = y^2 + c$

$x^2 - y^2 = c$

hyperbolae



(5) $F(x,y) = e^x \underset{\sim}{i} + e^{-x} \underset{\sim}{j}$

$$\frac{dx}{e^x} = \frac{dy}{e^{-x}} = dy\, e^x$$

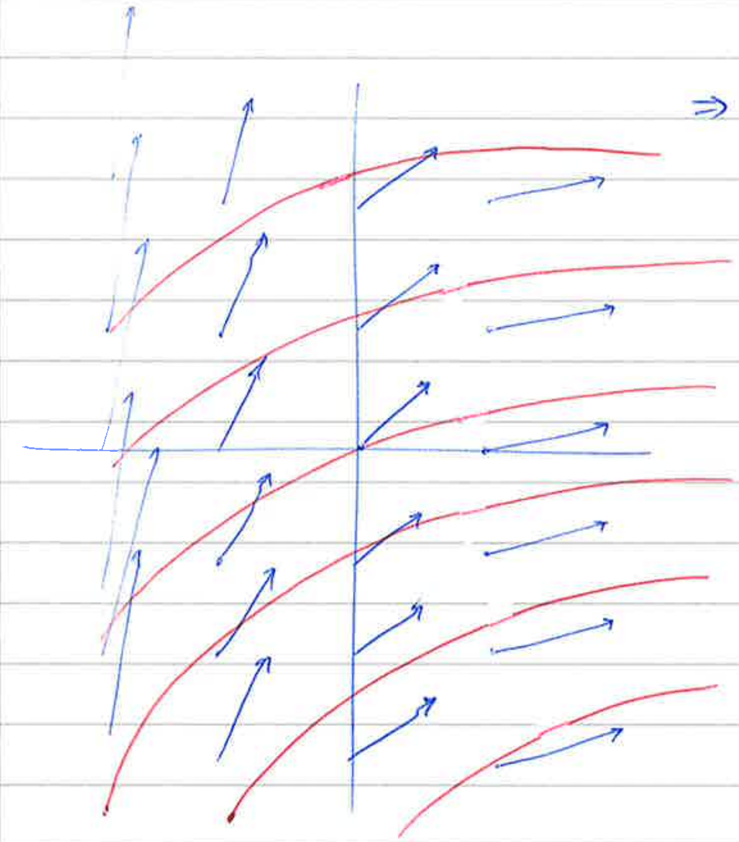
$\Rightarrow$

$$e^{-2x}\, dx = dy$$

$\Rightarrow$ $-\frac{1}{2} e^{-2x} = y + c$

$y = -\frac{1}{2} e^{-2x} + c'$.

graph of $-\frac{1}{2} e^{-2x}$ shifted up & down by constants.



(7) $\underset{\sim}{F}(x,y) = \underset{\sim}{\nabla} \ln(x^2+y^2)$, $\underset{\sim}{F}$ is conservative with potential $\phi = \ln(x^2+y^2)$ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

$$= \frac{2x}{x^2+y^2} \underset{\sim}{i} + \frac{2y}{x^2+y^2} \underset{\sim}{j}$$

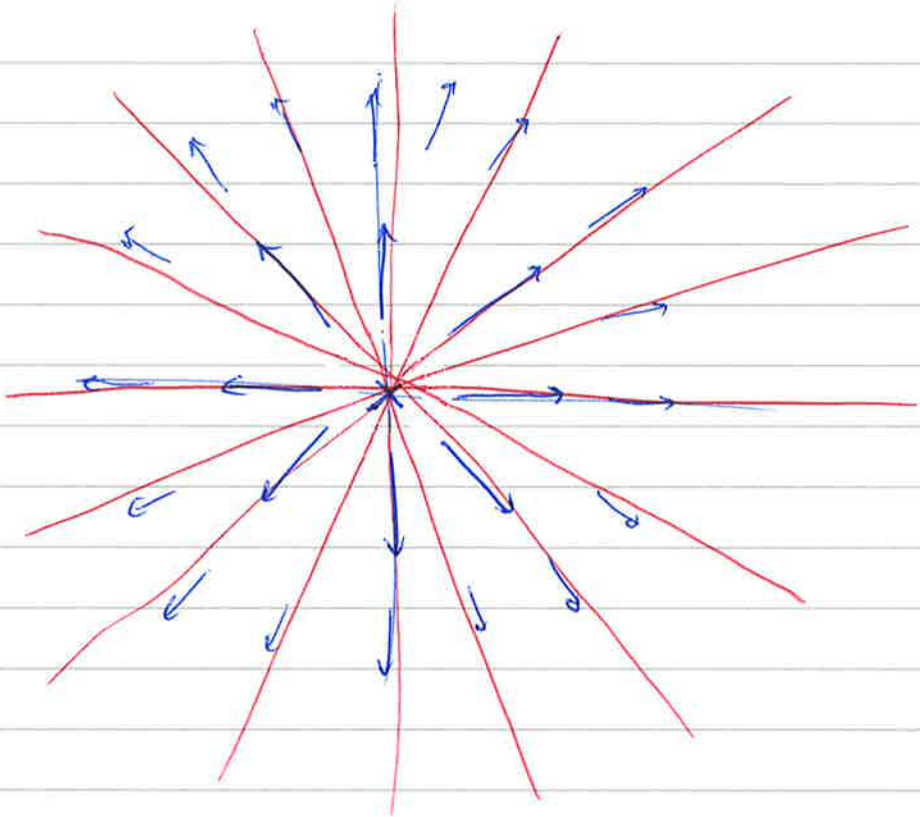
$$= 2 \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^2} = \frac{2 \hat{\underset{\sim}{r}}}{|\underset{\sim}{r}|}.$$

$\Rightarrow$ field lines are perpendicular to level curves of ϕ. (which are circles).

$\Rightarrow$ field lines are rays from origin / lines through origin.

$$\frac{dx}{\frac{2x}{x^2+y^2}} = \frac{dy}{2y/_{x^2+y^2}} \Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + \ln c$$

$\Rightarrow$ $x = cy$ as expected.



15.2 ex 2. $\underset{\sim}{F}(x,y,z) = y\underset{\sim}{i} + x\underset{\sim}{j} + z^2\underset{\sim}{k}$

$$\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}$$

$\Rightarrow$ $\underset{\sim}{F}$ could be conservative (it didn't fail the necessary condition to be conservative).
Need to find a potential to be sure.

$$\underset{\sim}{F} = \underset{\sim}{\nabla}\phi \Rightarrow \frac{\partial \phi}{\partial x} \overset{①}{=} y \qquad \partial\phi/\partial y \overset{②}{=} x \qquad \frac{\partial \phi}{\partial z} \overset{③}{=} z^2$$

$$\frac{\partial \phi}{\partial z} \overset{③}{=} z^2 \Rightarrow \phi(x,y,z) = \tfrac{1}{3} z^3 + c_1(x,y)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial c_1}{\partial x}(x,y) \overset{①}{=} y \quad \Rightarrow c_1(x,y) = xy + c_2(x)$$

$$\Rightarrow \phi(x,y,z) = \tfrac{1}{3} z^3 + xy + c_2(x) \Rightarrow \frac{\partial \phi}{\partial y} = x + \frac{\partial c_2(x)}{\partial x} \overset{②}{=} x \Rightarrow c_2 = \text{const.}$$

$$\Rightarrow \phi(x,y,z) = xy + \tfrac{1}{3} z^3 + c_2.$$

4. $\underset{\sim}{F}(x,y) = \dfrac{x\underset{\sim}{i} + y\underset{\sim}{j}}{x^2+y^2}$ defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial F_2}{\partial x} = \frac{y(-1)\cdot 2x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{x(-1)\cdot 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = \frac{\partial F_2}{\partial x}$$

So $\underset{\sim}{F}$ could be conservative. Try to find ϕ.

$$\frac{\partial \phi}{\partial x} \overset{①}{=} \frac{x}{x^2+y^2} \qquad \frac{\partial \phi}{\partial y} \overset{②}{=} \frac{y}{x^2+y^2}$$

$$\overset{①}{\Rightarrow} \phi(x,y) = \frac{1}{2}\ln(x^2+y^2) + c_1(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} + \frac{\partial c_1(y)}{\partial y} \overset{②}{=} \frac{y}{x^2+y^2} \Rightarrow c_1 = \text{constant}$$

$$\Rightarrow \phi(x,y) = \frac{1}{2}\ln(x^2+y^2) + c_1$$

$$= \ln\left(\sqrt{x^2+y^2}\right) + c_1 .$$

So $\underset{\sim}{F}$ was conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$.

5.

$$\underset{\sim}{F}(x,y,z) = (2xy - z^2)\underset{\sim}{i} + (2yz + x^2)\underset{\sim}{j} - (2zx - y^2)\underset{\sim}{k}$$

$$\frac{\partial F_1}{\partial y} = 2x = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = 2y = \frac{\partial F_3}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = -2z = \frac{\partial F_3}{\partial x},$$ $\Rightarrow \underset{\sim}{F}$ could be conservative. (can't rule it out!)

$$\frac{\partial \phi}{\partial x} \overset{①}{=} 2xy - z^2 \qquad \frac{\partial \phi}{\partial y} \overset{②}{=} 2yz + x^2 \qquad \frac{\partial \phi}{\partial z} \overset{③}{=} -2zx + y^2$$

$$② \Rightarrow \phi(x,y,z) = y^2 z + x^2 y + c_1(x,z)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xy + \frac{\partial c_1}{\partial x}(x,z) \overset{①}{=} 2xy - z^2$$

$$\Rightarrow \frac{\partial c_1}{\partial x} = -z^2 \Rightarrow c_1 = -z^2 x + c_2(z)$$

$$\phi = y^2 z + x^2 y - z^2 x + c_2(z)$$

$$\frac{\partial \phi}{\partial z} = y^2 - 2zx + \frac{\partial c_2}{\partial z} \overset{③}{=} -2zx + y^2 \Rightarrow c_2 = \text{const.}$$

$$\Rightarrow \phi(x,y,z) = y^2 z + x^2 y - z^2 x + c_2.$$

$\underset{\sim}{F}$ is conservative with potential ϕ.

6. $\underline{F}(x,y,z) = e^{x^2+y^2+z^2}(xz\underset{\sim}{i} + yz\underset{\sim}{j} + xy\underset{\sim}{k})$.

$$\frac{\partial F_1}{\partial y} = 2y e^{x^2+y^2+z^2} x z = \frac{\partial F_2}{\partial x}$$

$$\frac{\partial F_1}{\partial z} = e^{x^2+y^2+z^2}(2xz^2 + x)$$

$$\frac{\partial F_3}{\partial x} = e^{x^2+y^2+z^2}(2x^2 y + y) \neq \frac{\partial F_1}{\partial z}$$

$\underset{\sim}{F}$ cannot be conservative; it failed the necessary condition for it to be conservative.

7. $\phi(\underset{\sim}{r}) = \dfrac{1}{|\underset{\sim}{r} - \underset{\sim}{r}_0|^2} = \dfrac{1}{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$

$$\frac{\partial \phi}{\partial x} = \frac{(-1) \cdot 2(x-x_0)}{|\underset{\sim}{r} - \underset{\sim}{r}_0|^4}$$

$$\frac{\partial \phi}{\partial y} = \frac{-2(y-y_0)}{|\underset{\sim}{r} - \underset{\sim}{r}_0|^4}$$

$$\frac{\partial \phi}{\partial z} = -\frac{2(z-z_0)}{|\underset{\sim}{r} - \underset{\sim}{r}_0|^4}$$

$$\Rightarrow \underset{\sim}{F} = \underset{\sim}{\nabla}\phi = \frac{-2(\underset{\sim}{r} - \underset{\sim}{r}_0)}{|\underset{\sim}{r} - \underset{\sim}{r}_0|^4}.$$

15.4. 1. $\underset{\sim}{F}(x,y) = xy\,\underset{\sim}{i} - x^2\,\underset{\sim}{j}$ along $y = x^2$ from $(0,0)$ to $(1,1)$.

$\underset{\sim}{r}(x) = x\,\underset{\sim}{i} + x^2\,\underset{\sim}{j}, \quad x \in [0,1].$

$d\underset{\sim}{r} = dx\,\underset{\sim}{i} + 2x\,dx\,\underset{\sim}{j}$

$\underset{\sim}{F}(x, x^2) = x^3\,\underset{\sim}{i} - x^2\,\underset{\sim}{j} \Rightarrow \underset{\sim}{F} \cdot d\underset{\sim}{r} = x^3 dx - 2x^3 dx$
$= -x^3 dx$

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = -\int_0^1 x^3\,dx = -1/4.$$

2. $\underset{\sim}{F}(x,y) = \cos x\,\underset{\sim}{i} - y\,\underset{\sim}{j}$ along $y = \sin x$, $(0,0)$ to $(\pi, 0)$.
$\underset{\sim}{F}(x, \sin x) = \cos x\,\underset{\sim}{i} - \sin x\,\underset{\sim}{j}$
$\underset{\sim}{r}(x) = x\,\underset{\sim}{i} + \sin x\,\underset{\sim}{j}, \quad x \in [0, \pi]$

$d\underset{\sim}{r} = dx\,\underset{\sim}{i} + \cos x\,dx\,\underset{\sim}{j} \Rightarrow \underset{\sim}{F} \cdot d\underset{\sim}{r} = \cos x\,dx - \sin x \cos x\,dx$

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{\pi} (\cos x - \sin x \cos x) \; dx$$

$$\sin x \cos x = 1/2 \; \sin 2x$$

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \sin x + 1/4 \cos 2x \Big|_0^{\pi}$$

$$= 1/4 \left[1 - 1 \right] = 0.$$

3. $\underline{F}(x,y,z) = y\underset{\sim}{i} + z\underline{j} - x\underline{k}$ along straight line from $(0,0,0)$ to $(1,1,1)$.

$\underline{F}(\underline{r}(t)) = t\underline{i} + t\underline{j} - t\underline{k}$

$\underset{\sim}{r}(t) = t(1,1,1) = t\underset{\sim}{i} + t\underline{j} + t\underline{k}$, $t \in [0,1]$.

$d\underset{\sim}{r} = dt\,(\underset{\sim}{i} + \underline{j} + \underline{k})$.

$$\underset{\sim}{F} \cdot d\underline{r} = (t + t - t)\,dt = t\,dt$$

$$\int_{\mathcal{C}} \underset{\sim}{F} \cdot d\underline{r} = \int_0^1 t\,dt = 1/2.$$

4. $\underline{F}(x,y,z) = z\underset{\sim}{i} - y\underline{j} + 2x\underline{k}$ along $x=t$, $y=t^2$, $z=t^3$ from $(0,0,0)$ to $(1,1,1)$.

$\underline{r}(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$

$d\underline{r} = (\underset{\sim}{i} + 2t\underline{j} + 3t^2\underline{k})\,dt$

$$\underset{\sim}{F}(\underline{r}(t)) = t^3\underset{\sim}{i} - t^2\underline{j} + 2t\underline{k}$$

$$\underset{\sim}{F} \cdot d\underline{r} = (t^3 - 2t^3 + 6t^3)\,dt = 5t^3\,dt$$

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_0^1 5t^3\,dt = 5/4.$$

7. $\underset{\sim}{F}(x,y,z) = (x+y)\underset{\sim}{i} + (x-z)\underset{\sim}{j} + (z-y)\underset{\sim}{k}$

along C from $(1,0,-1)$ to $(0,-2,3)$.

This question only makes sense if $\underset{\sim}{F}$ is conservative, otherwise $\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r}$ depends on C, not just on the endpoints.

Find a potential ϕ for $\underset{\sim}{F}$:

$$\frac{\partial \phi}{\partial x} = x+y \qquad \frac{\partial \phi}{\partial y} = x-z \qquad \frac{\partial \phi}{\partial z} = z-y$$

$$\phi(x,y,z) = \tfrac{1}{2}x^2 + xy + c_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial c_1}{\partial y} = x - z \Rightarrow c_1 = -zy + c_2(z)$$

$$\frac{\partial \phi}{\partial z} = -y + \frac{\partial c_2}{\partial z} = z - y \Rightarrow c_2 = \tfrac{1}{2}z^2 + c_3.$$

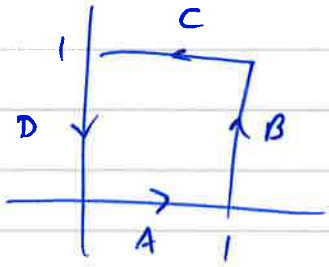
$$\phi(x,y,z) = \tfrac{1}{2}x^2 + xy - zy + \tfrac{1}{2}z^2 + c_3.$$

Free to choose $c_3 = 0$. (Doesn't affect answer.)

Then $\int_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \phi(0,-2,3) - \phi(1,0,-1)$

$$= 6 + \tfrac{1}{2}\cdot 9 - \left(\tfrac{1}{2} + \tfrac{1}{2}\right) = \frac{19}{2}.$$

8. $\oint_C x^2y^2\,dx + x^3y\,dy$



A: $y=0$, $\Rightarrow dy=0$ $\quad \int_0^1 0 \cdot dx = 0$

B: $x=1$, $dx=0$ $\quad \int_0^1 1^3 \cdot y\,dy = \frac{1}{2}$

C: $y=1$, $dy=0$ $\quad \int_1^0 x^2 \cdot 1^2\,dx = -\frac{1}{3}$

D: $x=0$, $dx=0$ $\quad \int_1^0 0^3 \cdot y\,dy = 0.$

$\oint_C = A+B+C+D = 1/6.$

15. (a) $\oint_C x\,dy$ around $x^2+y^2=a^2$.

Use polar coords: $x=a\cos\theta$ $\quad y=a\sin\theta$
$dx=-a\sin\theta\,d\theta$ $\quad dy=a\cos\theta\,d\theta$

$$\oint_C x\,dy = \int_0^{2\pi} a^2\cos^2\theta\,d\theta = \frac{1}{2}a^2\int_0^{2\pi}(\cos 2\theta+1)\,d\theta$$

$\left[\cos^2\theta = \frac{1}{2}(\cos 2\theta+1)\right]$

$$= \frac{1}{2}a^2\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{2\pi} = \frac{1}{2}a^2[2\pi] = \pi a^2.$$

(b) $\oint_{C} y\,dx = -\int_0^{2\pi} a^2 \sin^2\theta \, d\theta = \frac{a^2}{2}\int_0^{2\pi} (\cos 2\theta - 1)\, d\theta$

$\left[\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \right]$

$= \frac{a^2}{2}\left[+\frac{1}{2}\sin 2\theta - \theta \right]_0^{2\pi} = -\pi a^2 .$