16.1 (1) $\underset{\sim}{F} = x\underset{\sim}{i} + y\underset{\sim}{j} = F_1\underset{\sim}{i} + F_2\underset{\sim}{j} + F_3\underset{\sim}{k}$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 0 = 2$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix}$$

$$= \underset{\sim}{i}(0-0) + \underset{\sim}{j}(0-0) + \underset{\sim}{k}(0-0) = \underset{\sim}{0}.$$

(2) $\underset{\sim}{F} = y\underset{\sim}{i} + x\underset{\sim}{j}$, $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0 + 0 + 0 = 0$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & 0 \end{vmatrix} = \underset{\sim}{i}(0) + \underset{\sim}{j}(0) + \underset{\sim}{k}(1-1) = \underset{\sim}{0}$$

(3) $\underset{\sim}{F} = y\underset{\sim}{i} + z\underset{\sim}{j} + x\underset{\sim}{k}$, $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0 + 0 + 0 = 0$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \underset{\sim}{i}(0-1) + \underset{\sim}{j}(0-1) + \underset{\sim}{k}(0-1) = -\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}.$$

(4) $\underset{\sim}{F} = yz\,\underset{\sim}{i} + xz\,\underset{\sim}{j} + xy\,\underset{\sim}{k}$

$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0 + 0 + 0 = 0$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \underset{\sim}{i}(x - x) + \underset{\sim}{j}(y - y) + \underset{\sim}{k}(z - z) = \underset{\sim}{0}$$

(5) $\underset{\sim}{F} = x\,\underset{\sim}{i} + x\,\underset{\sim}{k}$ $\qquad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 1 + 0 + 0 = 1$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & 0 & x \end{vmatrix} = \underset{\sim}{i}(0 - 0) + \underset{\sim}{j}(0 - 1) + \underset{\sim}{k}(0) = -\underset{\sim}{j}$$

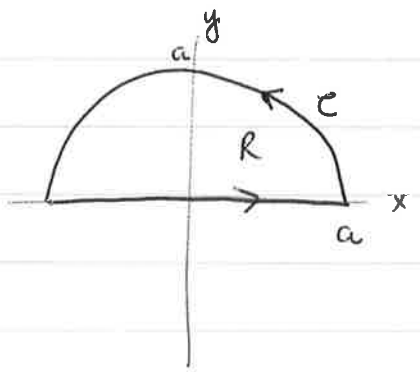
(6) $\underset{\sim}{F} = xy^2\,\underset{\sim}{i} - yz^2\,\underset{\sim}{j} + zx^2\,\underset{\sim}{k}$

$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = y^2 - z^2 + x^2$

$$\underset{\sim}{\nabla} \times \underset{\sim}{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & -yz^2 & zx^2 \end{vmatrix} = \underset{\sim}{i}(0 + 2yz) + \underset{\sim}{j}(0 - 2zx) + \underset{\sim}{k}(0 - 2xy)$$

$$= 2yz\,\underset{\sim}{i} - 2zx\,\underset{\sim}{j} - 2xy\,\underset{\sim}{k}.$$

16.3 (1) $\oint_C (\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy$



$= \oint_C F_1\,dx + F_2\,dy$

$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$

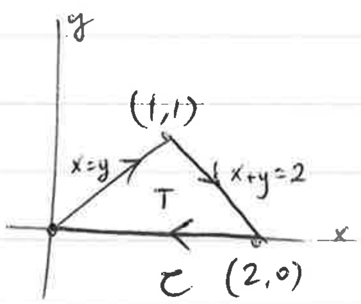
$= \iint_R (2 - 6y)\,dx\,dy$

$= \int_0^\pi d\theta \int_0^a (2 - 6r\sin\theta)\,r\,dr = \int_0^\pi d\theta\, (r^2 - 2r^3\sin\theta)\Big|_0^a$

$= \int_0^\pi a^2 - 2a^3\sin\theta\,d\theta = \pi a^2 + 2a^3\cos\theta\Big|_0^\pi = \pi a^2 - 4a^3$

(2) $\oint_C (x^2 - xy)\,dx + (xy - y^2)\,dy$

due to clockwise orientation on C



$= -\iint_T y - (-x)\,dx\,dy$

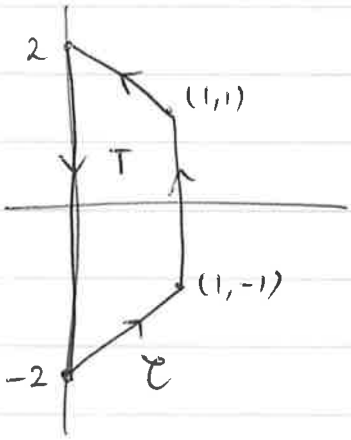
$= -\int_0^1 dy \int_y^{2-y} x + y\,dx$

$= -\int_0^1 dy \left(\frac{x^2}{2} + yx\right)\Big|_{x=y}^{2-y} = -\int_0^1 dy \left[\frac{(2-y)^2}{2} + y(2-y) - \frac{y^2}{2} - y^2\right]$

$= -\int_0^1 2 - 2y + \frac{y^2}{2} + 2y - y^2 - \frac{y^2}{2} - y^2\,dy = \int_0^1 -2 + 2y^2\,dy$

$= -2 + 2/3 = -4/3$

(3) $\oint_{\mathcal{C}} (x\sin(y^2) - y^2)\,dx + (x^2 y\cos(y^2) + 3x)\,dy$



$$= \iint_T 2xy\cos(y^2) + 3 - (2xy\cos(y^2) - 2y)\;dx\,dy$$

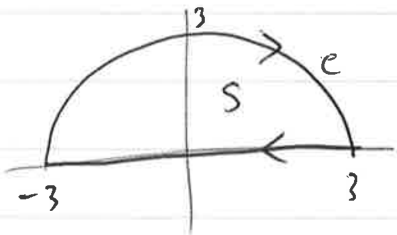
$$= \iint_T 3 + 2y\;dx\,dy$$

Due to symmetry of T about the x-axis (when $y \leftrightarrow -y$), the odd function $2y$ integrates to 0.

$$= 3\iint_T dx\,dy = 3\cdot \text{area}(T) = 3\cdot 1\cdot 3 = 9.$$

(4) $\oint_{\mathcal{C}} x^2 y\,dx - xy^2\,dy = -\iint_S -y^2 - x^2\,dx\,dy = \iint_S x^2+y^2\,dx\,dy$

clockwise on $\mathcal{C}$



$$= \int_0^{\pi} d\theta \int_0^3 r^2\, r\,dr = \pi\cdot\frac{81}{4}$$

(5) $\underset{\sim}{r} = a\cos^3 t\,\underset{\sim}{i} + b\sin^3 t\,\underset{\sim}{j}$, $0 \le t \le 2\pi$. Area $= \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos^3 t\; b\,3\sin^2 t\cos t + a\,b\sin^3 t\; 3\cos^2 t\sin t)\,dt$$

$$= \frac{3ab}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt = \frac{3ab}{8}\int_0^{2\pi}\sin^2(2t)\,dt = \frac{3ab}{16}\int_0^{2\pi} 1 - \cos 4t\,dt$$

$$= \frac{3ab}{16}\left[2\pi - 0\right] = 3\pi ab/8.$$

16.4 (1) $\underset{\sim}{F} = x\underset{\sim}{i} - 2y\underset{\sim}{j} + 4z\underset{\sim}{k}$ $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 1 - 2 + 4 = 3$

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_B \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = 3 \iiint_B dV = 3 \cdot \mathrm{vol}B = 3 \cdot \frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

(2) $\underset{\sim}{F} = ye^z \underset{\sim}{i} + x^2 e^z \underset{\sim}{j} + xy\underset{\sim}{k}$, $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 0 + 0 + 0 = 0$

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_B 0\, dV = 0.$$

(3) $\underset{\sim}{F} = (x^2 + y^2)\underset{\sim}{i} + (y^2 - z^2)\underset{\sim}{j} + z\underset{\sim}{k}$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 2x + 2y + 1$$

due to symmetry

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_B 2x + 2y + 1\, dV = 0 + 0 + \mathrm{vol}(B) = \frac{4}{3}\pi a^3$$

(4) $\underset{\sim}{F} = x^3 \underset{\sim}{i} + 3yz^2 \underset{\sim}{j} + (3y^2 z + x^2)\underset{\sim}{k}$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 3x^2 + 3z^2 + 3y^2 = 3R^2$$

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}}\, dS = \iiint_B 3R^2\, dV = 3\int_0^\pi d\phi \int_0^{2\pi} d\theta \int_0^a R^2 \; R^2 \sin\phi\, dR$$

$$= 3 \cdot 2 \cdot 2\pi \cdot \frac{a^5}{5} = \frac{12\pi a^5}{5}$$

(5) $\underset{\sim}{F} = x^2 \underset{\sim}{i} + y^2 \underset{\sim}{j} + z^2 \underset{\sim}{k}$ $\quad \underset{\sim}{\nabla} \cdot \underset{\sim}{F} = 2x + 2y + 2z = 2(x+y+z)$.

$(x-2)^2 + y^2 + (z-3)^2 \le 9$. Ball centred at $(2, 0, 3)$.

$$\oiint_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = \iiint_D \underset{\sim}{\nabla} \cdot \underset{\sim}{F}\, dV = 2 \iiint_D x+y+z\, dV. = I$$

Let $X = x-2$, $Y = y$, $Z = z-3$,
$dX = dx \quad dY = dy \quad dZ = dz$

$$I = 2 \iiint_D X+2 + Y + Z + 3 \; dX\, dY\, dZ$$

$$= 2 \iiint_D 5 + X + Y + Z \; dX\, dY\, dZ = 10 \operatorname{vol}(D) + 0 + 0 + 0$$ due to symmetry of D about X, Y, Z axes.

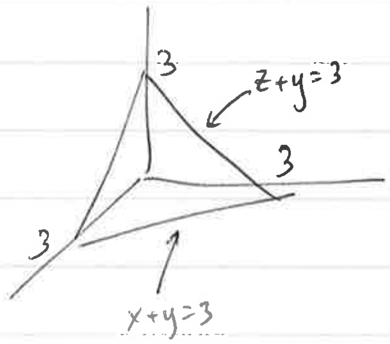
$$= 10 \cdot \tfrac{4}{3} \pi \cdot 3^3 = 360\pi$$

(6) $x^2 + y^2 + 4(z-1)^2 \le 4$ $\quad$ Let $Z = z-1$, $dZ = dz$

$$\oiint_S \underset{\sim}{F} \cdot \hat{\underset{\sim}{N}}\, dS = 2 \iiint_D x+y+z \; dx\,dy\,dz = 2 \iiint_D x+y+Z+1 \; dx\,dy\,dZ$$

$$= 2 \operatorname{vol}(D) = 2 \cdot \tfrac{4}{3} \pi \cdot 2 \cdot 2 \cdot 1 = \frac{32\pi}{3}$$

(7) $T: x+y+z \leq 3, \quad x \geq 0, \ y \geq 0, \ z \geq 0$



$$\iiint_T x \, dx\,dy\,dz = \int_0^3 dx \int_0^{3-x} dy \int_0^{3-x-y} x \, dz$$

$$= \int_0^3 dx \int_0^{3-x} dy \, \left(3x - x^2 - xy\right)$$

$$= \int_0^3 dx \ \ 3x(3-x) - x^2(3-x) - \frac{x}{2}(3-x)^2$$

$$= \int_0^3 9x - 3x^2 - 3x^2 + x^3 - \frac{9x}{2} + 3x^2 - \frac{x^3}{2} \, dx$$

$$= \int_0^3 \frac{x^3}{2} - 3x^2 + \frac{9x}{2} \, dx = \frac{81}{8} - 27 + \frac{81}{4}$$

$$= 27/8$$

So $\displaystyle\oiint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}} \, dS = 2 \iiint_T x+y+z \, dV = 6 \iiint_T x \, dV$ by symmetry under exchange of variables

$$= 81/4.$$

(8) $x^2+y^2 \leq 2y \Rightarrow x^2 + (y-1)^2 \leq 1, \qquad 0 \leq z \leq 4.$

$Y = y-1, \qquad Z = z-2$

$$\oiint_S \underset{\sim}{F} \cdot \underset{\sim}{\hat{N}} \, dS = 2 \iiint_R x+y+z \, dV = 2 \iiint_R x + Y + 1 + Z + 2 \, dx\,dY\,dZ$$

$$= 6 \iiint_R dx\,dY\,dZ = 6 \operatorname{vol}(R) = 6 \cdot \pi \cdot 1^2 \cdot 4 = 24\pi$$

all other terms vanish due to symmetry in x, Y, Z coords.