**Solution: MPE 340 Reservoir Simulation, Introduction**
**DATE: May 8, 2009**

**Problem 1**

a)

Different quantities:

$\rho$ density, $\phi$ porosity, $q$ source term
$\vec{v} = (v^x, v^y, v^z)$ velocity field
$\nabla \cdot \vec{v} = \partial_x v^x + \partial_y v^y + \partial_z v^z$

Darcy's law:

$\vec{v} = -\frac{[k]}{\mu}(\nabla p - \rho g \nabla d) = -\frac{[k]}{\mu}(\nabla p - \rho g \vec{k})$

$\mu$ viscosity, $p$ pressure, $g$ gravity constant, $d$ distance function in vertical direction; for z-axis parallel to vertical direction: $d(x, y, z) = z$.

$$[k] = \begin{pmatrix} k^x & 0 & 0 \\ 0 & k^y & 0 \\ 0 & 0 & k^z \end{pmatrix}$$

Resulting equation:

$$\nabla \cdot \Big(\rho \frac{[k]}{\mu}(\nabla p - \rho g \vec{k})\Big) + q = \frac{\partial}{\partial t}(\rho\phi).$$

Simplifying assumptions:
1D horizontal domain, constant $k^x$, $\mu$, and $\phi$.

b)

Let $\rho(p) = \frac{m}{V(p)}$. Then

$$\frac{d\rho(p)}{dp} = m\frac{d}{dp}\Big(\frac{1}{V(p)}\Big) = -\frac{m}{V^2}\frac{dV(p)}{dp} = cV\frac{m}{V^2} = c\rho,$$

where we have used $-cV = \frac{dV}{dp}$.

c)

First, we have

$$\rho c \nabla p(\rho) = \rho c \frac{dp}{d\rho}\nabla\rho.$$

Next, we use that $1 = \frac{dp(\rho)}{dp} = \frac{dp}{d\rho}\frac{d\rho}{dp}$ to conclude that

$$\rho c \nabla p(\rho) = \frac{\rho c}{\frac{d\rho}{dp}}\nabla\rho = \nabla\rho,$$

where we have used the result from b). Inserting this last relation (one-dimensional version) in Eq.(1) we get

$$\frac{k}{\mu}\frac{\partial}{\partial x}\Big(\rho\frac{1}{\rho c}\frac{\partial\rho}{\partial x}\Big) + q = \phi\frac{\partial\rho}{\partial t}.$$

Consequently, we obtain the desired density equation with coefficient $\kappa = \frac{k}{c\mu\phi}$.

**Problem 2**

a)

Assumptions:

1D horizontal flow, immiscible flow, incompressible fluids,
$\phi$ and $k$ are constant,
capillary pressure is neglected
reservoir is initially filled with oil
constant injection rate
existence of unique solution

$$f(S) = \frac{\lambda_w(S)}{\lambda_w(S)+\lambda_o(S)} = \frac{k_{rw}(S)}{k_{rw}(S)+Mk_{ro}(S)}, \qquad M = \frac{\mu_w}{\mu_o}, \qquad S = \text{water saturation.}$$

b)

Height $S_f$ of water front:

$$f'(S_f) = \frac{f(S_f)}{S_f} \quad \Rightarrow \quad S_f \approx 0.6 \qquad \text{(from Fig. 1).}$$

Front velocity $V$:

$$V = f'(S_f) = \frac{f(S_f)}{S_f} \approx \frac{0.88}{0.6} = 1.467, \qquad \text{since } f(0.6) \approx 0.88 \qquad \text{(from Fig. 1).}$$

Position of front at time $T = 0.4$:

$$x_f = VT = 1.467 \cdot 0.4 = 0.587.$$

Saturation $S_1$ located at $x_1 = 0.2$:

$$x_1 = f'(S_1)T \quad \Rightarrow \quad f'(S_1) = 0.5 \quad \Rightarrow \quad S_1 \approx 0.70 \qquad \text{(from Fig. 2).}$$

Saturation $S_2$ located at $x_2 = 0.4$:

$$x_2 = f'(S_2)T \quad \Rightarrow \quad f'(S_2) = 1 \quad \Rightarrow \quad S_2 \approx 0.64 \qquad \text{(from Fig. 2).}$$

c)

$T_b$ is given by

$$1 = VT_b \quad \Rightarrow \quad T_b \approx \frac{1}{1.467} = 0.68.$$

For $T < T_b$ we have the following oil recovery:

$$R = \frac{\text{volume of injected water}}{\text{volume of initial oil in place}} = A(T),$$

where $A(T)$ is the area limited by the solution $S(x,T)$ and the axis. Consequently,

$$\begin{aligned} R = A(T) &= x_f S_f + \int_{S^f}^{1} T f'(S)\, dS \\ &= f'(S_f)S_f T + T(f(1) - f(S_f)) = f(S_f)T + T(1 - f(S_f)) = T, \end{aligned}$$

since $f'(S_f)S_f = f(S_f)$.

For $T > T_b$ we have the same expression $R = A(T)$, but $A(T)$ is now given by

$$R = A(T) = S^*(T) \cdot 1 + \int_{S^*(T)}^{1} f'(S)T\, dS = S^*(T) + T(1 - f(S^*(T))),$$

where $S^*(T)$ now satisfy $1 = f'(S^*(T))T$. Consequently, for the case of b) with $T = 1$ we get

$$1 = f'(S^*(1)) \quad \Rightarrow \quad S^*(1) \approx 0.64 \qquad \text{(from Fig. 2)}$$

This implies,

$$R(T = 1) = 0.64 + 1(1 - f(0.64)) \approx 0.64 + 1 - 0.93 = 0.71 \qquad \text{since } f(0.64) \approx 0.93 \qquad \text{(from Fig. 1).}$$

**Problem 3**

a)

Main assumptions:

- 3 phases: water, oil, gas
- 3 components: water, oil, gas
- no phase transition between water and hydrocarbons
- a part of the gas component can be dissolved in oil (and flows together with the oil component in the oil phase)
- all of the oil component is in the oil phase
- constant temperature

b)

Different mass components that fill pore space:

1 water component in water phase: $\rho_w S_w$
1 oil component in oil phase: $\rho_o S_o$
1 gas component in oil phase: $\rho_{dg} S_o$
1 gas component in gas phase: $\rho_g S_g$

Continuity equations, respectively, for the water component, the oil component, and the two gas components:

$$\nabla \cdot (\rho_w \vec{v_w}) = -\frac{\partial}{\partial t}(\phi \rho_w S_w) + q_w$$

$$\nabla \cdot (\rho_o \vec{v_o}) = -\frac{\partial}{\partial t}(\phi \rho_o S_o) + q_o$$

$$\nabla \cdot (\rho_{dg} \vec{v_o} + \rho_g \vec{v_g}) = -\frac{\partial}{\partial t}(\phi \rho_{dg} S_o + \phi \rho_g S_g) + q_g$$

$\vec{v_l}$, $l = w, o, g$ (velocity fields)
$S_l$, $l = w, o, g$ (saturations) such that $S_w + S_o + S_g = 1$
$\phi$ porosity

c)

For the densities at standard conditions we have:

$$\rho_l^s = \frac{m_l}{[V_l]_{ST}}, \qquad l = w, o, g, \qquad \rho_{dg}^s = \frac{m_{dg}}{[V_{dg}]_{ST}} = \rho_g^s$$

and volume factors $B_l$ and gas-oil solution ratio $R_s$ are defined as

$$R_s = \frac{[V_{dg}]_{ST}}{[V_o]_{ST}}, \qquad B_l = \frac{[V_l]_{RC}}{[V_l]_{ST}}, \qquad l = w, o, g$$

Using this, we have

$$\rho_w = \frac{m_w}{[V_w]_{RC}} = \rho_w^s \frac{[V_w]_{ST}}{[V_w]_{RC}} = \frac{\rho_w^s}{B_w}$$
$$\rho_o = \frac{m_o}{[V_o]_{RC}} = \rho_o^s \frac{[V_o]_{ST}}{[V_o]_{RC}} = \frac{\rho_o^s}{B_o}$$
$$\rho_g = \frac{m_g}{[V_g]_{RC}} = \rho_g^s \frac{[V_g]_{ST}}{[V_g]_{RC}} = \frac{\rho_g^s}{B_g}$$
$$\rho_{dg} = \frac{m_{dg}}{[V_o]_{RC}} = \rho_g^s \frac{[V_{dg}]_{ST}}{[V_o]_{RC}} \cdot \frac{[V_o]_{ST}}{[V_o]_{ST}} = \frac{\rho_g^s}{B_o} R_s$$

Inserting these relations in equations of b), dividing by the constant density $\rho_l^s$, we arrive at

$$\nabla \cdot (\frac{1}{B_w} \vec{v_w}) = -\frac{\partial}{\partial t}(\phi \frac{S_w}{B_w}) + \frac{q_w}{\rho_w^s}$$
$$\nabla \cdot (\frac{1}{B_o} \vec{v_o}) = -\frac{\partial}{\partial t}(\phi \frac{S_o}{B_o}) + \frac{q_o}{\rho_o^s}$$
$$\nabla \cdot (\frac{R_s}{B_o} \vec{v_o} + \frac{1}{B_g} \vec{v_g}) = -\frac{\partial}{\partial t}(\phi \frac{R_s S_o}{B_o} + \phi \frac{S_g}{B_g}) + \frac{q_g}{\rho_g^s}$$

**Problem 4**
a)
Water-oil system:

$$k_{rw}^{2p}: \quad S_{wr} \leq S_w \leq 1, \qquad S' = \frac{S_w - S_{wr}}{1 - S_{wr}} \in [0,1], \qquad k_{rw}^{2p}(S_w) = k_{rw}^t(S') = (S')^3$$
$$k_{row}^{2p}: \quad S_{wr} \leq S_w \leq 1 - S_{owr}, \qquad S' = \frac{S_w - S_{wr}}{1 - S_{wr} - S_{owr}} \in [0,1], \qquad k_{row}^{2p}(S_w) = k_{row}^t(S') = (1 - S')^2$$

Gas-oil system:

$$k_{rg}^{2p}: \quad S_{gr} \leq S_g \leq 1 - S_{wr}, \qquad S' = \frac{S_g - S_{gr}}{1 - S_{wr} - S_{gr}} \in [0,1], \qquad k_{rg}^{2p}(S_g) = k_{rg}^t(S') = (S')^2$$
$$k_{rog}^{2p}: \quad 0 \leq S_g \leq 1 - S_{wr} - S_{ogr}, \qquad S' = \frac{S_g}{1 - S_{wr} - S_{ogr}} \in [0,1], \qquad k_{rog}^{2p}(S_w) = k_{row}^t(S') = (1 - S')^3$$

b)
Three-phase relative permeability curves for water, gas, and oil are obtained as follows:

$$k_{rw}^{3p}(S_w) = k_{rw}^{2p}(S_w)$$
$$k_{rg}^{3p}(S_g) = k_{rg}^{2p}(S_g)$$
$$k_{ro}^{3p}(S_w, S_g) = \text{interpolation of } k_{row}^{2p}(S_w) \text{ and } k_{rog}^{2p}(S_g)$$

c)

$$
\begin{aligned}
k_{rw}(S_w = 0.5) &= \left(\frac{0.5 - S_{wr}}{1 - S_{wr}}\right)^3 = \left(\frac{0.35}{0.85}\right)^3 = \left(\frac{7}{17}\right)^3 = 0.4117^3 = 0.069 \\
k_{rg}(S_g = 0.3) &= \left(\frac{0.3 - S_{gr}}{1 - S_{wr} - S_{gr}}\right)^2 = \left(\frac{0.10}{0.65}\right)^2 = \left(\frac{2}{13}\right)^2 = 0.154^2 = 0.024 \\
k_{row}^{2p}(S_w = 0.5) &= \left(1 - \frac{0.5 - S_{wr}}{1 - S_{wr} - S_{owr}}\right)^2 = (1 - 0.35/0.75)^2 = 0.284 \\
k_{rog}^{2p}(S_g = 0.3) &= \left(1 - \frac{0.3}{1 - S_{wr} - S_{ogr}}\right)^3 = (1 - 0.3/0.6)^3 = 0.5^3 = 0.125 \\
k_{ro}(S_w = 0.5, S_g = 0.3) &= \frac{0.35}{0.65} k_{row}^{2p}(S_w = 0.5) + \frac{0.30}{0.65} k_{rog}^{2p}(S_g = 0.3) \\
&= 0.538 \cdot 0.284 + 0.462 \cdot 0.125 = 0.153 + 0.058 = 0.21
\end{aligned}
$$

**Problem 5**

The following equation is given,

$$
\frac{Ckb_r}{\mu}\frac{\partial^2 p}{\partial x^2} + q = \phi c_f b_r \frac{\partial p}{\partial t}. \tag{1}
$$

a)

Looking at block no. $i$ we let $A$ denote the boundary plane with the neighbor block $i-1$ in the negative $x$-direction and $B$ the boundary plane with the neighbor $i+1$ in the positive $x$-direction. To simplify we define $Y$ by

$$
Y = \frac{Ckb_r}{\mu}\frac{\partial p}{\partial x},
$$

and (1) is written

$$
\frac{\partial Y}{\partial x} + q = \phi c_f b_r \frac{\partial p}{\partial t}. \tag{2}
$$

The finite difference scheme is used to discretize equation (2), the right-hand-side in the $x$-direction and the left-hand-side in time, and

$$
\frac{Y_i|_B - Y_i|_A}{\Delta x} + q_i = \phi c_f b_r \frac{p_i(t_{n+1}) - p_i(t_n)}{\Delta t}. \tag{3}
$$

Furthermore

$$
Y_i|_B \approx \frac{Ckb_r}{\mu}\frac{p_{i+1}(t) - p_i(t)}{\Delta x},
$$

and correspondingly for the other term,

$$
Y_i|_A \approx \frac{Ckb_r}{\mu}\frac{p_i(t) - p_{i-1}(t)}{\Delta x}.
$$

These two expressions gives respectively the volum flow of oil in from the right and left to block $i$ and equation 3 sets the difference equal to the change in oil volume per time by expansion or injection in the block. In these expressions for the volume rate into block $i$ we have to specify at which time level the pressures are chosen. The most common choice in modern industrial simulators is to chose $t$ at new time $t_{n+1}$, the so-called implicit formulation.

Inserted into equation 3 we get

$$\frac{Ckb_r}{\mu(\Delta x)^2}[p_{i+1}(t_{n+1}) - p_i(t_{n+1})] \tag{4}$$
$$-\frac{Ckb_r}{\mu(\Delta x)^2}[p_i(t_{n+1}) - p_{i-1}(t_{n+1})] + q_i = \phi c_f b_r (p_i(t_{n+1}) - p_i(t_n))/\Delta t.$$

b)

Let us now set $T = Ckb_r/\mu(\Delta x)^2$ and simplify the notation by setting $p_k \equiv p_k(t_{n+1})$ for arbitrary $k$. The equation rearranged is

$$Tp_{i-1} - (2T + \phi c_f b_r/\Delta t)p_i + Tp_{i+1} = -(\phi c_f b_r p_i(t_n)/\Delta t + q_i).$$

The end blocks require special attention. We will assume a closed reservoir with no flow across the left and right boundaries: $Y_1|_A = 0$ and $Y_l|_B = 0$, when $N$ denotes the last block. For the first block the equation reads

$$-(T + \phi c_f b_r/\Delta t)p_1 + Tp_2 = -(\phi c_f b_r p_1(t_n)/\Delta t + q_1),$$

and for the last block $N$

$$Tp_{N-1} - (T + \phi c_f b_r/\Delta t)p_N = -(\phi c_f b_r p_N(t_n)/\Delta t + q_N).$$

Let $m$ be 1 if the block number $i$ is 1 or $N$ and 2 otherwise. Then we can define

$$\begin{aligned} W &= mT + \phi c_f b_r/\Delta t \\ a_i &= T/W \\ c_i &= T/W \\ d_i &= (\phi c_f b_r p_i(t_n)/\Delta t + q_i/W, \end{aligned}$$

and get equation 3 in the following numerical form,

$$\boxed{a_i p_{i-1} - p_i + c_i p_{i+1} = -d_i.} \tag{5}$$