Solution: Modeling part.

Task 2.

(a) We have

$$u_t = \frac{\partial}{\partial t}u_0(x-ct) = u_0'(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) = u_0'(x-ct)\cdot(-c)$$
$$u_x = \frac{\partial}{\partial x}u_0(x-ct) = u_0'(x-ct)\cdot\frac{\partial}{\partial x}(x-ct) = u_0'(x-ct).$$

Hence,

$$u_t + cu_x = -cu_0'(x-ct) + cu_0'(x-ct) = 0.$$

The solution can also be written as

$$u(x,t) = u_0(x-ct) = \begin{cases} 1, & 0.4 \le x-ct \le 0.6; \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$u(x,t) = u_0(x-ct) = \begin{cases} 1, & 0.4+ct \le x \le 0.6+ct; \\ 0, & \text{otherwise.} \end{cases}$$

The solutions is obtained by moving the initial box towards right with a speed equal to c. The boundary condition at $x=1$ can only hold before the box reaches $x=1$. The box will reach the right boundary $x=1$ when

$$0.6 + cT^b = 1, \qquad \Rightarrow \qquad cT^b = 0.4.$$

(b) Upwind flux:

$$U^n_{j+1/2} = u^n_j, \qquad U^n_{j-1/2} = u^n_{j-1}.$$

Resulting scheme

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n), \qquad \lambda = \frac{\Delta t}{\Delta x}.$$

See the text document **Project-Transport1** (Section 1.4) for the calculations that show that

$$\sum_{j=1}^{M}|u_j^{n+1}| \le \sum_{j=1}^{M}|u_j^n|,$$

if $0 \le \lambda \le 1$.

(c) We have

$$(\phi\rho)_t + (\rho u)_x = 0,$$

with $\phi(p)$ porosity, $\rho(p)$ density, and u Darcy velocity.
Darcy's law

$$u = -\frac{k}{\mu}p_x,$$

inserted gives

$$(\phi(p)\rho(p))_t = (\rho(p)\frac{k}{\mu}p_x)_x.$$

Weakly compressible rock:

$$\phi(p) = \phi_0 e^{c_r[p-p_0]},$$

linear version

$$\phi(p) = \phi_0(1 + c_r[p-p_0]).$$

We assume that $\rho(p) = \rho_0$ (constant). Plugging these two relations into equation gives

$$(\phi_0(1+c_r[p-p_0]))_t = (\frac{k}{\mu}p_x)_x.$$

That is

$$\phi_0 c_r p_t = \frac{k}{\mu}(p_x)_x,$$

if k, μ are constant. This gives $p_t = \kappa p_{xx}$ with

$$\kappa = \frac{k}{\mu \phi_0 c_r}.$$

(d) Continuous $p_0(x)$ if

$$2 + a = b + 1 \qquad \Rightarrow \qquad a + 1 = b.$$

Physical interpretation: Fluid in $[0, 1/2]$ will flow in left direction, whereas fluid in $[1/2, 1]$ will towards right as the pressure will decrease.
Stationary solution:

$$p_{xx} = 0 \qquad \Rightarrow \qquad p(x) = Cx + D.$$

Boundary conditions give that

$$p(x) = [b - a]x + a.$$

Stationary fluid velocity

$$u = -\frac{k}{\mu}p_x = -\frac{k}{\mu}(b - a).$$

(e) We refer to Section 1.5 in the note **Project-Pressure1**.

Task 3. This exercise deals with the Black-Oil Model (BOM).

(a) Main assumptions:
- 3 phases: water, oil, gas
- 3 components: water, oil, gas
- no phase transition between water and hydrocarbons
- a part of the gas component can be dissolved in oil (and flows together with the oil component in the oil phase)
- all of the oil component is in the oil phase
- constant temperature

Different mass components that fill pore space:
1 water component in water phase: $\rho_w S_w$
1 oil component in oil phase: $\rho_o S_o$
1 gas component in oil phase: $\rho_{dg} S_o$
1 gas component in gas phase: $\rho_g S_g$

Continuity equations, respectively, for the water component, the oil component, and the two gas components:

$$\nabla \cdot (\rho_w \vec{v_w}) = -\frac{\partial}{\partial t}(\phi \rho_w S_w) + q_w$$
$$\nabla \cdot (\rho_o \vec{v_o}) = -\frac{\partial}{\partial t}(\phi \rho_o S_o) + q_o$$
$$\nabla \cdot (\rho_{dg} \vec{v_o} + \rho_g \vec{v_g}) = -\frac{\partial}{\partial t}(\phi \rho_{dg} S_o + \phi \rho_g S_g) + q_g$$

$\vec{v_l}$, $l = w, o, g$ (velocity fields)
S_l, $l = w, o, g$ (saturations) such that $S_w + S_o + S_g = 1$
ϕ porosity

(b) For the densities at standard conditions we have:

$$\rho_l^s = \frac{m_l}{[V_l]_{ST}}, \qquad l = w, o, g, \qquad \rho_{dg}^s = \frac{m_{dg}}{[V_{dg}]_{ST}} = \rho_g^s$$

and volume factors B_l and gas-oil solution ratio R_s are defined as

$$R_s = \frac{[V_{dg}]_{ST}}{[V_o]_{ST}}, \qquad B_l = \frac{[V_l]_{RC}}{[V_l]_{ST}}, \qquad l = w, o, g$$

Using this, we have

$$\begin{aligned}
\rho_w &= \frac{m_w}{[V_w]_{RC}} = \rho_w^s \frac{[V_w]_{ST}}{[V_w]_{RC}} = \frac{\rho_w^s}{B_w} \\
\rho_o &= \frac{m_o}{[V_o]_{RC}} = \rho_o^s \frac{[V_o]_{ST}}{[V_o]_{RC}} = \frac{\rho_o^s}{B_o} \\
\rho_g &= \frac{m_g}{[V_g]_{RC}} = \rho_g^s \frac{[V_g]_{ST}}{[V_g]_{RC}} = \frac{\rho_g^s}{B_g} \\
\rho_{dg} &= \frac{m_{dg}}{[V_o]_{RC}} = \rho_g^s \frac{[V_{dg}]_{ST}}{[V_o]_{RC}} \cdot \frac{[V_o]_{ST}}{[V_o]_{ST}} = \frac{\rho_g^s}{B_o} R_s
\end{aligned}$$

Inserting these relations in equations of a), dividing by the constant density ρ_l^s, we arrive at

$$\begin{aligned}
\nabla \cdot \left(\frac{1}{B_w} \vec{v_w}\right) &= -\frac{\partial}{\partial t}\left(\phi \frac{S_w}{B_w}\right) + \frac{q_w}{\rho_w^s} \\
\nabla \cdot \left(\frac{1}{B_o} \vec{v_o}\right) &= -\frac{\partial}{\partial t}\left(\phi \frac{S_o}{B_o}\right) + \frac{q_o}{\rho_o^s} \\
\nabla \cdot \left(\frac{R_s}{B_o} \vec{v_o} + \frac{1}{B_g} \vec{v_g}\right) &= -\frac{\partial}{\partial t}\left(\phi \frac{R_s S_o}{B_o} + \phi \frac{S_g}{B_g}\right) + \frac{q_g}{\rho_g^s}
\end{aligned}$$

Task 4.

(a) See the text document **Theory-B-L-base** for details. Mass conservation gives the relation

$$f'(s^*) = \frac{f(s^*)}{s^*}.$$

(b) From Fig. we see that $s^* \approx 0.75$. Speed $V = f'(s^*) = f(s^*)/s^* \approx 0.9/0.75 = 1.2$. At time $T = 0.5$, position of front is

$$x^* = VT = 1.2 \cdot 0.5 = 0.6.$$

Behind front:

$$\begin{aligned}
s &= 0.8 \qquad f'(0.8) \approx 0.75 \qquad \Rightarrow \qquad x_s = 0.75 \cdot 0.5 = 0.375. \\
s &= 0.9 \qquad f'(0.9) \approx 0.125 \qquad \Rightarrow \qquad x_s = 0.125 \cdot 0.5 = 0.06.
\end{aligned}$$

(c) The fractional flow f for $M = 0.5$ lies above the one corresponding to $M = 4$. This gives rise to a lower front height and a larger slope, i.e., a faster front. See Fig. 1.

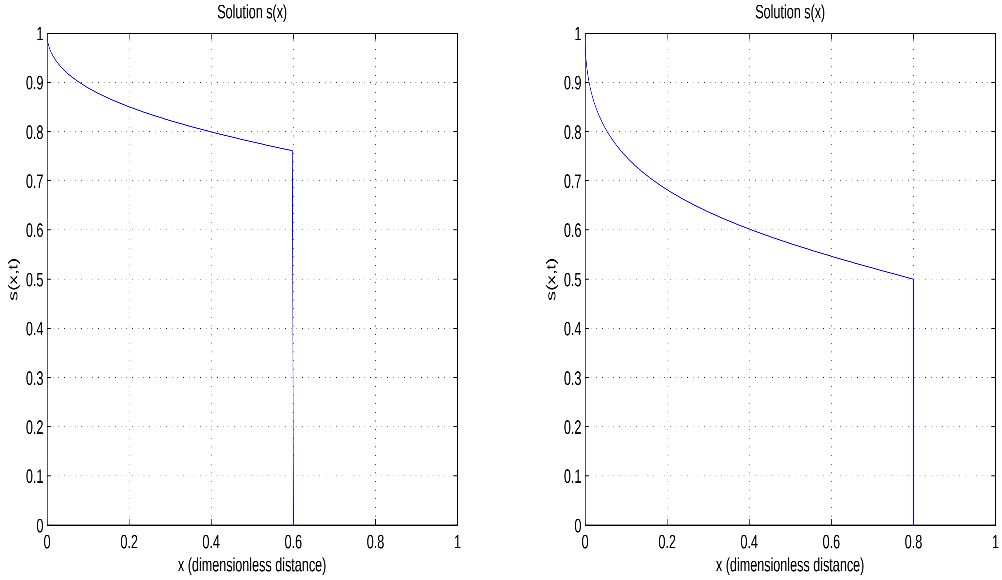

FIGURE 1. **Left:** Solution in (b). **Right:** Solution in (c).