Solution Pet 510 Fall 2013: Part A

Problem 1.

(a) Consider the linear transport equation

$$(*)\qquad u_t + a(x)u_x = b(x,u), \qquad x \in \mathbb{R} = (-\infty, +\infty), \qquad u(x, t = 0) = u_0(x) = \phi(x).$$

- Transport effect represented by the term $a(x)u_x$: If $a(x) < 0$, then a transport from right towards left. If $a(x) > 0$, transport from left towards right.
Impact from the source term $b(x,u)$: If $b > 0$ this will lead to a growth of u, whereas $b < 0$ will imply that u is reduced?

(b) Let $a(x) = x$ and $b(x,u) = 0$ in (*). Characteristic $X(t)$ is given

$$\frac{d}{dt}X(t) = X(t), \qquad X(t=0) = x_0,$$

which gives the solution $X(t) = x_0 e^t$. Moreover,

$$\frac{d}{dt}u(X(t),t) = u_x\frac{dX}{dt} + u_t = u_x X(t) + u_t = 0,$$

since u is a solution of $u_t + xu_x = 0$ and satisfies $u_t + X(t)u_x = 0$ along $X(t)$, i.e.

$$u(X(t),t) = u(x_0, t=0) = \phi(x_0) = \phi(X(t)e^{-t}).$$

Conclusion: $u(x,t) = \phi(xe^{-t})$.
Check:
(i) We see that $u(x, t = 0) = \phi(xe^0) = \phi(x)$, thus, initial data is satisfied.
(ii) Moreover, we see that

$$u_t = \phi'(xe^{-t})(xe^{-t})_t = \phi'(xe^{-t})x \cdot (-1) \cdot e^{-t}$$

and

$$u_x = \phi'(xe^{-t})(xe^{-t})_x = \phi'(xe^{-t})e^{-t}$$

so, clearly, $u_t + xu_x = 0$.

(c) Let $a(x) = x$ and $b(x,u) = u$ in (*). Characteristic $X(t)$ is given as above. Moreover,

$$\frac{d}{dt}u(X(t),t) = u_x\frac{dX}{dt} + u_t = u_x X(t) + u_t = u(X(t),t),$$

since u satisfies $u_t + xu_x = u$. From this we get

$$\int_{u(x_0,t=0)}^{u(X(t),t)} \frac{1}{u}\,du = \int_0^t dt$$

which gives us

$$\ln(u(X(t),t)) - \ln(\phi(x_0)) = t, \qquad \text{or} \qquad u(X(t),t) = \phi(x_0)e^t = \phi(X(t)e^{-t})e^t.$$

Conclusion: $u(x,t) = \phi(xe^{-t})e^t$
Check:

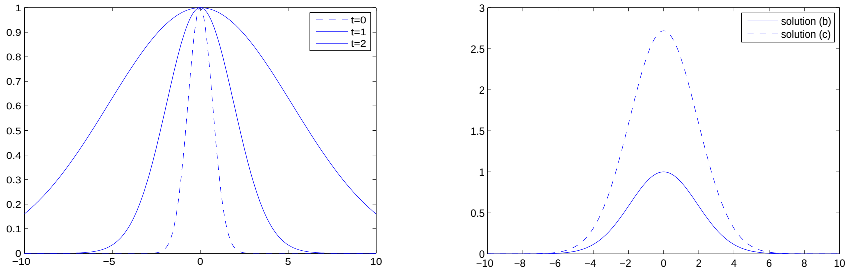


FIGURE 1. **Left:** Plot of u in (b) at times $t = 1$ and $t = 2$. **Right:** Comparison of u from (b) and (c) computed at time $t = 1$.

(i) We see that $u(x, t = 0) = \phi(xe^0)e^0 = \phi(x)$, thus, initial data is satisfied.
(ii) Moreover, we see that

$$\begin{aligned} u_t &= \phi'(xe^{-t})(xe^{-t})_t \cdot e^t + \phi(xe^{-t}) \cdot (e^t)_t \\ &= -\phi'(xe^{-t})xe^{-t} \cdot e^t + \phi(xe^{-t})e^t = -\phi'(xe^{-t})x + \phi(xe^{-t})e^t \end{aligned}$$

and

$$u_x = \phi'(xe^{-t})(xe^{-t})_x \cdot e^t = \phi'(xe^{-t})e^{-t} \cdot e^t = \phi'(xe^{-t})$$

Clearly, $u_t + xu_x = \phi(xe^{-t})e^t = u$

(d) Now we choose $\phi(x) = \exp(-x^2)$.
- Solution of (b) at times $t = 1$ and $t = 2$ is shown in Fig. 1 (left).
- Solution of (b) and (c) at time $t = 1$ are shown in Fig. 1 (right). Main difference is the growth in u for solution computed in part (c).

(e) $U^n_{j+1/2} = u^n_j$ and $U^n_{j-1/2} = u^n_{j-1}$. This gives the scheme $u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}(u^n_j - u^n_{j-1})$.

(f) Estimate for the discrete scheme:

$$|u^{n+1}_j| = |u^n_j(1 - \lambda) + \lambda u^n_{j-1}| \le |u^n_j(1 - \lambda)| + |\lambda u^n_{j-1}| = (1 - \lambda)|u^n_j| + \lambda|u^n_{j-1}|$$

Here we first have used the triangle inequality $|a+b| \le |a|+|b|$. Then we have assumed that $0 \le \lambda \le 1$ in order to ensure that $(1 - \lambda) \ge 0$ and $\lambda \ge 0$. Now we sum over all cells from 1 to M

$$\begin{aligned}\sum_{j=1}^{M}|u_j^{n+1}| &\le (1-\lambda)\sum_{j=1}^{M}|u_j^n| + \lambda\sum_{j=1}^{M}|u_{j-1}^n| \\ &= (1-\lambda)\sum_{j=1}^{M}|u_j^n| + \lambda\sum_{j=0}^{M-1}|u_j^n| \qquad \text{(shift of index in last sum)} \\ &\le (1-\lambda)\sum_{j=1}^{M}|u_j^n| + \lambda\sum_{j=1}^{M}|u_j^n| \qquad \text{(add } |u_M^n| \text{ in the last sum)} \\ &= \sum_{j=1}^{M}|u_j^n|.\end{aligned}$$

Hence, the conclusion is that $\sum_{j=1}^{M}|u_j^{n+1}| \le \sum_{j=1}^{M}|u_j^n| \le \ldots \le \sum_{j=1}^{M}|u_j^0|$,
Condition on the discretization parameters Δt and Δx: $0 \le \lambda = \frac{\Delta t}{\Delta x} \le 1$

Problem 2.

(a) Mass balance

$$(\phi\rho)_t + (\rho u)_x = 0,$$

where ϕ, ρ, and u are porosity, fluid density, and fluid velocity (Darcy velocity). Darcy's law:

$$u = -\frac{k}{\mu}p_x$$

This gives

$$(\phi(p)\rho(p))_t = (\frac{k}{\mu}\rho(p)p_x) = \frac{k}{\mu}(\rho(p)p_x)_x$$

Using assumptions on ϕ and ρ we get

$$\rho\phi_0[1 + c_r(p - p_0)]_t = \rho\phi_0 c_r p_t = \frac{k}{\mu}\rho p_{xx}.$$

This gives us

$$p_t = \kappa p_{xx}, \qquad \kappa = \frac{k}{\mu\phi_0 c_r}$$

(b) Verify that

$$p(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2}d\theta$$

is a solution of $p_t = p_{xx}$. Note that $F(x,t) = \int_{-\infty}^{b(x,t)} G'(\theta)\theta = G(b(x,t)) - G(-\infty)$. Consequently,

$$F(x,t)_t = G'(b)b_t, \qquad F(x,t)_x = G'(b)b_x.$$

Using this with $b = \frac{x}{2\sqrt{t}}$ and $G'(\theta) = e^{-\theta^2}$ we get

$$p_t = \frac{1}{\sqrt{\pi}} e^{-b^2} b_t = \frac{1}{\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \left[\frac{x}{2\sqrt{t}}\right]_t = \frac{x}{2\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \cdot \frac{-1}{2} t^{-3/2}$$

and

$$p_x = \frac{1}{\sqrt{\pi}} e^{-b^2} b_x = \frac{1}{\sqrt{\pi}} e^{-[\frac{x}{2\sqrt{t}}]^2} \left[\frac{x}{2\sqrt{t}}\right]_x = \frac{1}{2\sqrt{\pi} t^{1/2}} e^{-[\frac{x}{2\sqrt{t}}]^2}$$

and

$$p_{xx} = \frac{1}{2\sqrt{\pi} t^{1/2}} \left[e^{-[\frac{x^2}{4t}]}\right]_x = -\frac{1}{2\sqrt{\pi} t^{1/2}} e^{-[\frac{x^2}{4t}]} \cdot [\frac{x^2}{4t}]_x == -\frac{x}{4\sqrt{\pi} t^{3/2}} e^{-[\frac{x^2}{4t}]}$$

Clearly, $p_t = p_{xx}$.
Initial condition:

$$x < 0: \qquad t \to 0^+ \Rightarrow \frac{x}{2\sqrt{t}} \to -\infty \Rightarrow p(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\infty} e^{-\theta^2} d\theta = 0$$

and

$$x > 0: \qquad t \to 0^+ \Rightarrow \frac{x}{2\sqrt{t}} \to +\infty \Rightarrow p(x,t) \to \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\theta^2} d\theta = 1$$

(c) Discrete scheme (explicit in time):

$$\frac{p_j^{n+1} - p_j^n}{\Delta t} = \frac{1}{\Delta x}([p_x]_{j+1/2}^n - [p_x]_{j-1/2}^n)$$

$$j = 1, \ldots, M-1: \qquad [p_x]_{j+1/2} = \frac{p_{j+1} - p_j}{\Delta x}$$

$$x = 0: \qquad [p_x]_{1/2} = \frac{p_1 - 0}{\Delta x/2}$$

$$x = 1: \qquad [p_x]_{M+1/2} = \frac{0 - p_M}{\Delta x/2}$$

Stability condition: $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$

(d) Estimate: We multiply the equation by p and get

$$p p_t = p_{xx} p$$

or

$$\frac{1}{2}(p^2)_t = (p_x p)_x - p_x p_x.$$

Next, we integrate in space from 0 to 1:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 p^2 \, dx = \int_0^1 (p_x p)_x \, dx - \int_0^1 (p_x)^2 \, dx.$$

For the first term on the right hand side we get:

$$\int_0^1 (p_x p)_x \, dx = p_x p|_{x=0}^{x=1} = 0,$$

using the boundary condition. Consequently, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 p^2\,dx = -\int_0^1 (p_x)^2\,dx \leq 0$$

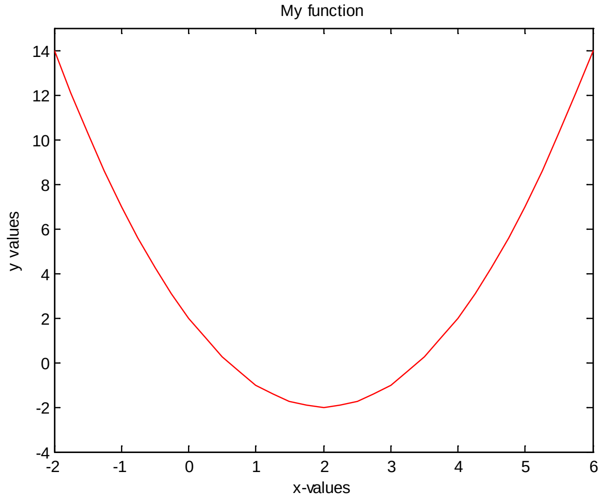
which gives the inequality $\frac{d}{dt}\int_0^1 p^2\,dx \leq 0$. We finally integrate in time over $[0, t]$.

Exam Part B – Solving Nonlinear Equations & Modeling of Well Flow

Exercise 3

a) Her kan det være mange varianter som kan være riktig.

```
a=-2;

b= 6;

dx = 1;

n = (b-a)/dx;

for i = 1:n+1

x(i)=a+(i-1)*dx;

y(i) =func(x(i));

end

plot(x,y);
```



b) Sette søkeintervall til feks [0,1]Forandre main til: a = 0, b = 1. Kravet er at funksjonen skal skifte fortegn i intervallet hvis vi skal klare å finne en løsning på f(x) = 0.

c) ftol sier noe om nøyaktighet på løsning. Vi er fornøyde med løsningen x3 hvis f(x3) < ftol. Hvis vi øker ftol vil vi få mer unøayktig løsning. Man kan og visualisere dette som en retangulær boks som roten må befinne seg inne og denne har høyde 2xftol.

d)Show how the bisection method works by filling out the table below (We assume ftol = 0.1):

Iteration	x1	x2	x3	f(x1)	f(x2)	f(x3)
1	2	4	3	-2	2	-1
2	3	4	3.5	-1	2	0,25
3	3	3.5	3.25	-1	0,25	-0,43
4	3.25	3.5	3.3750	-0,4374	0,25	-0.1094
5	3.3750	3.5	3.4375	-0.1094	0,25	0.0664

Exercise 4

a)I inch = 0.0254 m. This gives ID = 0.127 m. OD = 0.2159 m.

$A = \frac{3.14}{4}(0.2159^2 - 0.127^2) = 0.0239m^2$. Convert velocity v to m/s.

$$v = 200\frac{ft}{\min} = \frac{200 \cdot 0.3048m}{60s} = 1.016\frac{m}{s}.$$

$$Q = v \cdot A = 1.016 \cdot 0.0239\frac{m^3}{s} = 0.0243\frac{m^3}{s} = 0.0243 \cdot 1000 \cdot 60\frac{l}{\min} = 1458\frac{l}{\min}.$$

b) Smaller cuttings-> need larger velocity to transport cuttings. Higher ROP ->need larger velocity to transport cuttings. Larger mudweight gives more buouancy and gives lower required velocity.

c) We needed to introduce a forloop starting at the beginning of the program and ending at the end of the program. Inside this loop we let angle increase

```
ang(i)=angstart+i*5  .
```

where we defined angstart = 50; among the other variables in the top part of the program, We then implement the correction factor and include this is as correction term in the vslip calculation:

```
CANG = 0.0342*ang(i)-0.000233*ang(i)^2-0.213

vslip = vslip*CZ*CMW*CANG;   % Include correction factors
```

Finally we take care of the calculated rate for each angle variation through the statement:

```
yrate(i)=Q
```

After ending the for loop, we just write plot(ang,yrate);

Exercise 5

a) ECD = (3500x1.6x0.0981+25)/(3500x0.0981)=1.67 sg
b) Use Boyles law with 1 bar at surface: V2=(V1xP1)/P2=4x550/1=2200 m3
c) Because we then can take into account that e.g. mud density and hydrostatic pressure will depend on pressure and temp downhole. Pressures will be calculated locally in each cell and the results will be summarized in a net pressure effect.
 (The student might also mention that the friction will be calculated more properly also since local variations are taken into account, but it is suff if they explain that mud depend on temp, pressure)

Exercise 6

a) A steady state flow describes constant flow . A transient model describes flow that changes in time
b) The gas density at surface will be 1 kg/m3 using the ideal gas law in App D. The mass flow rate will be: $M = Q \cdot \rho = \frac{40}{60}\frac{m^3}{s} \cdot 1\frac{kg}{m^3} = 0.667 kg/s$ which will be constant all over in the well- The gas density at bottom is $\rho_g = \frac{200 \cdot 100000}{316 \cdot 316} = 200 kg/m3$. The flow rate will be:
$$Q = \frac{M}{\rho} = \frac{0.667}{200} = 0.00333 m3/s$$
c) The well is first discretized into cells/nodes. We willk now the massflowrates at bottom and the surface outlet pressure on top. We start by guessing for the bottomhole pressure. Then we use the closure laws in AppD to calc the remaining flow varibles in node 1. The the three conservation laws in combination with closure laws are used to find the flow varibles in node 2. Then we proceed to the next node and so on until we reach the outlet.The calculated pressure at outlet must be equal to Psurf. If not, the BHP must be guessed again. This reduced to finding the zero of the following function f(puess,bottom) =Pcalc, outlet-Psurf,given. This will be found using e.g. the bisection method. (important to understand the last part also here). The method is sometimes called the shooting technique.