Problem 1.

(a) **Solution:**

$$u(x,t) = v(x,\hat{t}), \qquad \hat{t} = \frac{t}{\varepsilon}$$

Then

$$u_t = \frac{\partial}{\partial t} v(x,\hat{t}) = \frac{\partial}{\partial \hat{t}} v(x,\hat{t}) \frac{\partial \hat{t}}{\partial t} = \frac{1}{\varepsilon} v_{\hat{t}}$$

and

$$u_x = \frac{\partial}{\partial x} v(x,\hat{t}) = v_x, \qquad u_{xx} = v_{xx}$$

Thus,

$$v_{\hat{t}} = \varepsilon u_t = u_{xx} = v_{xx}$$

Note also that $v(x,\hat{t}=0) = u(x,t=0)$, hence $v(x,\hat{t})$ must have a solution of the form (***):

$$v(x,\hat{t}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{\hat{t}}}} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x\sqrt{\varepsilon}}{2\sqrt{t}}} e^{-\theta^2} d\theta = u(x,t)$$

(b) **Solution:**

$$u(x,t) = v(\hat{x},t), \qquad \hat{x} = x\sqrt{\varepsilon}$$

Then

$$u_t = \frac{\partial}{\partial t} v(\hat{x},t) = v_t$$

and

$$u_x = \frac{\partial}{\partial x} v(\hat{x},t) = \frac{\partial}{\partial \hat{x}} v(\hat{x},t) \frac{\partial \hat{x}}{\partial x} = v_{\hat{x}}\sqrt{\varepsilon}, \qquad u_{xx} = v_{\hat{x}\hat{x}}\varepsilon$$

Thus,

$$v_t = u_t = \frac{1}{\varepsilon} u_{xx} = v_{\hat{x}\hat{x}}$$

Note also that $v(\hat{x},t=0) = u(x,t=0)$, hence $v(\hat{x},t)$ must have a solution of the form (***):

$$v(\hat{x},t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\hat{x}}{2\sqrt{t}}} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x\sqrt{\varepsilon}}{2\sqrt{t}}} e^{-\theta^2} d\theta = u(x,t)$$

Choosing a "small" ε implies that the upper limit $\frac{x\sqrt{\varepsilon}}{2\sqrt{t}}$ becomes "small" so that a large x must be chosen to give the same value $u(x,t)$. This corresponds to a strong smearing of the initial front at $x=0$. Choosing a "large" ε has the opposite effect, i.e., a weaker smearing of the front.

(c) **Solution:**
Let

$$y = \frac{x}{\sqrt{t}}, \qquad u(x,t) = v(y)$$

Then

$$u_t = \frac{\partial}{\partial t} v(y) = \frac{dv}{dy}(y) \frac{\partial y}{\partial t} = v'(y)(-\frac{1}{2}) \frac{x}{t^{3/2}}$$

Similarly,

$$u_x = v'(y)\frac{\partial y}{\partial x} = v'(y)\frac{1}{t^{1/2}}, \qquad u_{xx} = v''(y)\frac{1}{t}$$

Hence,

$$u_t = u_{xx} \qquad \text{correponds to} \qquad v'(y)(-\frac{1}{2})\frac{x}{t^{3/2}} = v''(y)\frac{1}{t}$$

which is the same as

$$0 = \frac{1}{2}v'(y)y + v''(y)$$

Since

$$\frac{x}{t^{1/2}} \to \begin{cases} -\infty, & x < 0; \\ +\infty, & x > 0 \end{cases}$$

when t goes towards 0 and $u(x,t)$ goes towards $u_0(x)$, then

$$v(-\infty) = u_0(x < 0) = 0, \qquad v(+\infty) = u_0(x > 0) = 1.$$

(d) **Solution:**

$$\begin{aligned} &u_t = u_{xx} + g(u), \qquad x \in (0,1) \\ &u(0,t) = u(1,t) = 0 \\ &u(x, t = 0) = u_0(x) \end{aligned}$$

We multiply equation by u which gives

$$\frac{1}{2}(u^2)_t = u_{xx}u + ug(u) = (u_x u)_x - u_x^2 + ug(u)$$

Integrating over $[0,1]$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx &= \int_0^1 (u_x u)_x dx - \int_0^1 u_x^2 dx + \int_0^1 ug(u)dx \\ &= (u_x u)\Big|_{x=0}^{x=1} - \int_0^1 u_x^2 + \int_0^1 ug(u)dx \\ &= 0 - \int_0^1 u_x^2 + \int_0^1 ug(u)dx \le \int_0^1 ug(u)dx \end{aligned}$$

If we assume that $ug(u) \le 0$, then we get

$$\frac{d}{dt}\int_0^1 u^2 dx \le 0.$$

An example is $g(u) = -u$ which gives $ug(u) = -u^2 \le 0$ for all u.

(e) **Solution:**

$$\begin{aligned} &p_t = \kappa p_{xx}, \qquad x \in (0,1), \qquad \kappa = \frac{k}{\phi_0(c + c_r)} \\ &p(0,t) = p(1,t) = 1 \\ &p(x, t = 0) = p_0(x) \end{aligned}$$

with initial data

$$(*) \qquad p_0(x) = \begin{cases} 1 - 2x, & 0 \le x < \frac{1}{2}; \\ 2x - 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

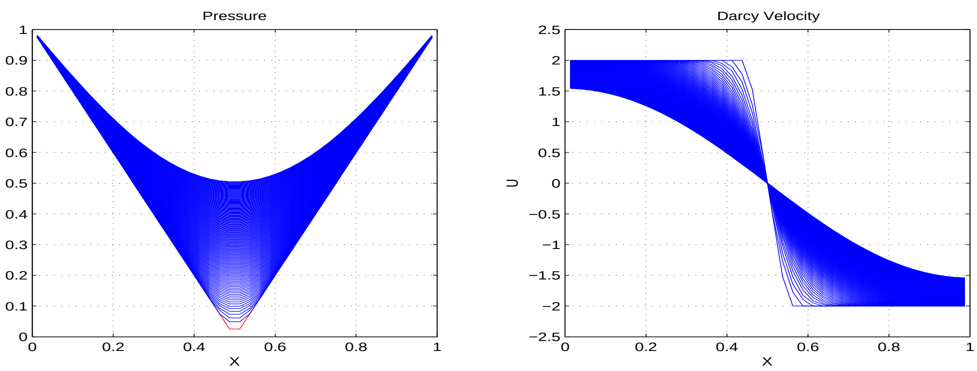

FIGURE 1. Curves show change in profiles from initial state towards stationary state.

Scheme:

$$\frac{P_j^{n+1} - P_j^n}{\Delta t} = \kappa \frac{1}{\Delta x}\Big([P_x]^n_{j+1/2} - [P_x]^n_{j-1/2}\Big), \qquad j = 1, \dots, M$$

where

$$[P_x]^n_{j+1/2} = \frac{P^n_{j+1} - P^n_j}{\Delta x}, \qquad j = 1, \dots, M-1$$

$$[P_x]^n_{1/2} = \frac{P^n_1 - 1}{\Delta x/2}$$

$$[P_x]^n_{M+1/2} = \frac{1 - P^n_M}{\Delta x/2}$$

Stability condition:

$$\mu = \frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

(f) **Solution:** Sketch of the initial data $p_0(x)$ and the solution $p(x,t)$ are shown in Fig. 1. See the right figure for plots of fluid velocity $u_0(x)$ and $u(x,t)$ based on Darcy law $u = -\frac{k}{\mu} p_x = -p_x$.
Corresponding to a decreasing pressure there is a fluid flow from left towards the center $x = 0.5$, whereas there is flow of fluid from right towards $x = 0.5$ associated with the increasing pressure. At $x = 0.5$ the fluid is stagnant.
Stationary solution: $p = 1$ and $u = 0$.

Problem 2. We consider a transport equation of the form

$$(B1) \qquad u_t + x u_x = b(x,t,u), \qquad x \in (-\infty, +\infty)$$

$$(B2) \qquad u(x,t) = u_0(x) = \exp(-x^2)$$

(a) Compute exact solution when $b(x,t,u) = -u$
- make a sketch of typical characteristics in the $x - t$ coordinate system

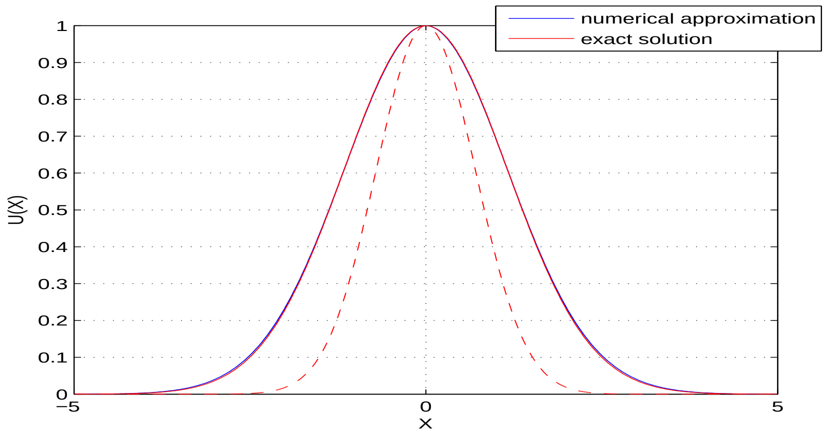

FIGURE 2. Numerical solution computed by using the scheme described in point (b) below which is compared with the exact solution at time $T = 0.5$

Solution:

$$\frac{dX}{dt} = X(t), \qquad X(t=0) = x_0 \qquad \Rightarrow \qquad X(t) = x_0 e^t$$

Thus,

$$\frac{du}{dt}(X(t),t) = -u(X(t),t) \qquad \Rightarrow \qquad \ln u(X(t),t) - \ln u_0(x_0) = -t$$
$$\Rightarrow \qquad u(X(t),t) = u_0(x_0)e^{-t}$$

Solution is

$$u(x,t) = u_0(xe^{-t}) \cdot e^{-t}.$$

The last term e^{-t} comes from the source term $b(u) = -u$.

(b) We now consider the model (B1) with $b = 0$. Consider a discretization of the spatial domain $[-5, 5]$. Assume that the domain is divided into $2M$ cells. It is assumed that $u_1^{n+1} = u_{2M}^{n+1} = 0$. Formulate a stable discrete scheme for cells $2, \ldots, 2M-1$.
Solution:

$$j = 2, \ldots, M: \qquad u_j^{n+1} = u_j^n - \lambda x_j (u_{j+1}^n - u_j^n) \qquad (\text{since } x_j < 0)$$
$$j = M+1, \ldots, 2M-1: \qquad u_j^{n+1} = u_j^n - \lambda x_j (u_j^n - u_{j-1}^n) \qquad (\text{since } x_j > 0)$$

(c) For the case considered above where the source term $b = 0$, show that the solution u of (B1) and (B2) satisfies the relation

$$\int_{-\infty}^{\infty} u(x,t)dx = e^t \int_{-\infty}^{\infty} u_0(x)dx.$$

Solution: Write equation in the form

$$u_t + (xu)_x - u = 0$$

and integrate in space over $(-\infty, \infty)$ and use that $(xu)(\pm\infty, t) = 0$. This gives the ODE

$$\frac{dv}{dt} = v, \qquad v = \int_{-\infty}^{\infty} u(x,t)dx$$

whose solution is

$$v(t) = v_0 e^t.$$

(d) Consider (B1) and (B2) again with $b(x,t,u) = -u + x$. Compute exact solution and verify that it satisfies (B1) and (B2).

Hint: The solution of an ODE of the form

$$(*) \qquad \frac{dv}{dt} + v = b(t) \qquad \text{is given by} \qquad v(t) = e^{-t}v_0 + e^{-t}\int_0^t b(s)e^s ds.$$

Solution: Using the characteristic found in (a) we have

$$\frac{du(X(t),t)}{dt} = -u(X(t),t) + X(t) = -u(X(t),t) + x_0 e^t$$

Hence, setting $v(t) = u(X(t),t)$ we have an ODE of the form (*) whose solution gives

$$\begin{aligned} u(X(t),t) &= u_0(x_0)e^{-t} + e^{-t}x_0\int_0^t e^{2s}ds \\ &= u_0(X(t)e^{-t})e^{-t} + e^{-t}X(t)e^{-t}\frac{1}{2}[e^{2t} - 1] \end{aligned}$$

This gives the solution

$$u(x,t) = u_0(xe^{-t})e^{-t} + \frac{1}{2}x[1 - e^{-2t}]$$

Check:

$$\begin{aligned} u_t &= -e^{-t}u_0 - xe^{-2t}u_0' + xe^{-2t}, \\ u_x &= e^{-2t}u_0' + \frac{1}{2}(1 - e^{-2t}) \end{aligned}$$

which gives

$$\begin{aligned} u_t + xu_x &= -e^{-t}u_0 - xe^{-2t}u_0' + xe^{-2t} + xe^{-2t}u_0' + \frac{1}{2}x(1 - e^{-2t}) \\ &= -e^{-t}u_0 + \frac{1}{2}x + \frac{1}{2}xe^{-2t} \\ &= -u + \frac{1}{2}x[1 - e^{-2t}] + \frac{1}{2}x + \frac{1}{2}xe^{-2t} = -u + x \end{aligned}$$