Problem 1.

(a) Key calculations are:

$$u_t + f'(u)u_x = 0$$

has characteristics given by

$$\frac{dX(t)}{dt} = f'(u(X(t),t)), \qquad X(t=0) = x_0, \qquad (*)$$

Hence,

$$\frac{d}{dt}u(X(t),t) = u_x\frac{dX}{dt} + u_t\frac{dt}{dt} = u_x f'(u) + u_t = 0$$

which implies that

$$u(X(t),t) = u(x_0, t=0) = \phi(x_0) \qquad (**).$$

We must find and expression for x_0. From (*) we get

$$\frac{dX(t)}{dt} = f'(u(X(t),t)) = f'(\phi(x_0)), \qquad X(t) = f'(\phi(x_0))t + x_0$$

Inserting in (**) we get

$$u(X(t),t) = \phi\Big(X(t) - f'(\phi(x_0))t\Big)$$

(b) We compute

$$u_x = \phi'(x_0)\frac{\partial}{\partial x}\Big(x - f'(u)t\Big) = \phi'(x_0)(1 - f''(u)u_x t) = \phi'(x_0)(1 - f''(\phi(x_0))u_x t) \qquad (b1)$$

From this we find the following expression for u_x:

$$u_x(1 + f''(\phi(x_0))\phi'(x_0)t) = \phi'(x_0)$$

or

$$u_x = \frac{\phi'(x_0)}{1 + f''(\phi(x_0))\phi'(x_0)t},$$

which blows up if

$$1 + f''(\phi(x_0))\phi'(x_0)t = 0$$

Since $f'' > 0$ this can only happen for some time $t > 0$ if $\phi'(x_0) > 0$.
We find

$$u_t = \phi'(x_0)\frac{\partial}{\partial t}\Big(x - f'(u)t\Big) = -\phi'(x_0)\Big(f''(u)u_t t + f'(u)\Big) \qquad (b2)$$

From (b1) and (b2) we get

$$\begin{aligned}
u_t + f'(u)u_x &= -\phi'(x_0)\Big(f''(u)u_t t + f'(u)\Big) + f'(u)\phi'(x_0)(1 - f''(\phi(x_0))u_x t) \\
&= -\phi'(x_0)f''(u)u_t t - f'(u)\phi'(x_0)f''(\phi(x_0))u_x t \\
&= -\phi'(x_0)f''(\phi(x_0))t[u_t + f'(u)u_x]
\end{aligned}$$

Hence,

$$[u_t + f'(u)u_x](1 + \phi'(x_0)f''(\phi(x_0))t) = 0.$$

Condition is then

$$1 + \phi'(x_0)f''(\phi(x_0))t \neq 0$$

(c) We refer to the note for the integral equality.
Motivation:
In the weak formulation derivatives have been over to the test function. Hence, we can include solutions that might contain discontinuities

(d) Consider (*) with $f(u) = \frac{1}{4}u^2$ and

$$\phi(x) = \begin{cases} 4, & 0 \le x < 1; \\ 0, & \text{otherwise} \end{cases}$$

RP-1 at $x = 0$:

$$v_t + f(v)_x = 0, \qquad f(v) = \frac{1}{4}v^2, \qquad f'(v) = \frac{1}{2}v \quad (\text{i.e., } (f')^{-1}(v) = 2v), \qquad v_0(x) = \begin{cases} 0, & x < 0; \\ 4, & x \ge 0; \end{cases}$$

Since we have in increasing jump, we cannot have a shock solution. We look for a rarefaction wave solution of the general form:

$$v(x/t) = \begin{cases} v_l, & x/t \le f'(v_l); \\ (f')^{-1}(x/t), & f'(v_l) < x/t < f'(v_r); \\ v_r, & f'(v_r) \le x/t. \end{cases}$$

Plugging in the values of $v_l = 0$, $v_r = 4$ we get

$$v(x/t) = \begin{cases} 0, & x/t \le 0; \\ 2(x/t), & 0 < x/t < 2; \\ 4, & 2 \le x/t. \end{cases} \tag{1}$$

RP-2 at $x = 1$: At $x = 1$ we have the following problem:

$$w_t + f(w)_x = 0, \qquad f(w) = \frac{1}{4}w^2, \qquad f'(w) = \frac{1}{2}w, \qquad w_0(x) = \begin{cases} 4, & x < 1; \\ 0, & x \ge 1; \end{cases}$$

This jump is decreasing, hence we can obtain an entropy satisfying solution by constructing a shock solution. The RH condition gives the speed

$$s = \frac{f(w_l) - f(w_r)}{w_l - w_r} = \frac{1/4 \cdot 4^2 - 1/4 \cdot 0^2}{4 - 0} = 1.$$

Clearly, the entropy condition is then satisfied since

$$f'(w_l) = 2 > s = 1 > 0 = f'(w_r).$$

Hence, the correct solution of this local Riemann problem is given by

$$w(x,t) = \begin{cases} w_l, & \text{if } x - 1 \le st; \\ w_r, & \text{if } x - 1 > st. \end{cases}$$

That is,

$$w(x,t) = \begin{cases} 4, & \text{if } x - 1 \le t; \\ 0, & \text{if } x - 1 > t. \end{cases} \tag{2}$$

Consequently, there is a left rarefaction wave and a right shock as described by the full solution

$$u(x,t) = \begin{cases} 0, & x \le 0; \\ 2(x/t), & 0 < x < 2t; \\ 4, & 2t \le x \le t + 1 \\ 0, & x > t + 1. \end{cases} \tag{3}$$

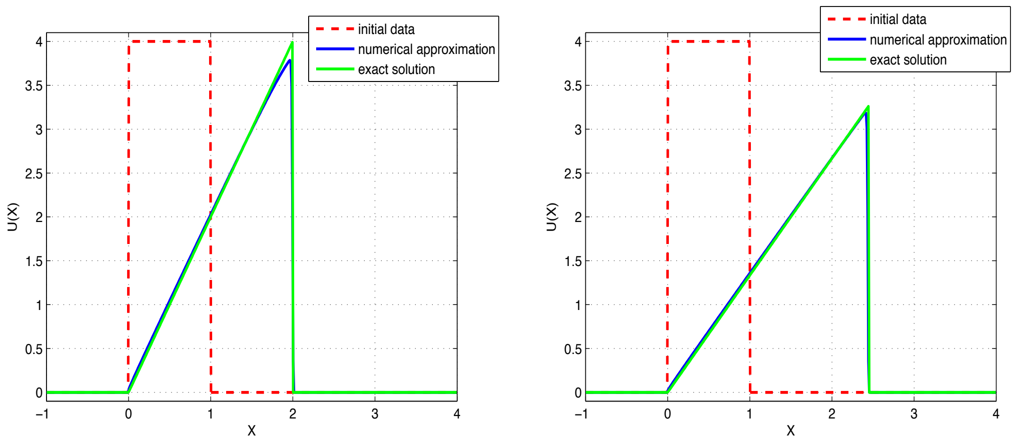

FIGURE 1. Left: Solution at time $T = 1$. Right: Solution at time $T = 1.5$.

At some time T^c the rarefaction wave catches up to the shock, this time is given as

$$2T^c = T^c + 1, \qquad \Rightarrow \qquad T^c = 1.$$

The solution (3) is valid for $t \in [0, T^c]$.

(e) We want to calculate the new shock $(u_l(t), 0)$ and its position $x_s(t)$ for times $t > T^c$. Firstly, the R-H condition gives the shock speed

$$s(t) = \frac{f(0) - f(u_l(t))}{0 - u_l(t)} = \frac{1}{4}u_l(t). \tag{4}$$

Secondly, the characteristic associated with $u_l(t)$ is given by

$$x_s(t) = f'(u_l(t))t. \tag{5}$$

How can we link these two relations?
The shock must be characterized by the fact that its speed $s(t)$ is equal to the speed of $u_l(t)$ along its characteristic $x_s(t)$. Thus,

$$s(t) = \frac{d}{dt}x_s(t).$$

Combining this relation with (4) and (5) gives that

$$\frac{1}{4}u_l(t) = \frac{d}{dt}[f'(u_l(t))t] = f''(u_l(t))u_l'(t)t + f'(u_l(t)) = \frac{1}{2}u_l'(t)t + \frac{1}{2}u_l(t).$$

That is

$$-\frac{1}{2}u_l(t) = \frac{du_l}{dt}t,$$

or

$$-\frac{1}{2}\frac{1}{t}dt = \frac{1}{u_l}du_l.$$

Integrating over $[T^c, t]$ and $[4, u_l(t)]$ gives

$$-\frac{1}{2}\int_{T^c}^{t} \frac{1}{t}dt = \int_4^{u_l} \frac{1}{u_l}du_l, \qquad t \geq T^c,$$

or

$$-\frac{1}{2}\ln(t)\Big|_{T^c}^{t} = \ln(u_l)\Big|_4^{u_l},$$

which implies

$$\ln\Big(\frac{T^c}{t}\Big)^{1/2} = \ln\Big(\frac{u_l}{4}\Big).$$

From this we get

$$u_l(t) = 4\Big(\frac{T^c}{t}\Big)^{1/2} = \frac{4}{t^{1/2}}, \qquad t \geq T^c = 1. \tag{6}$$

The corresponding position $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2}u_l(t)t = 2\sqrt{t}, \qquad t \geq T^c = 1. \tag{7}$$

The resulting solution for $t \geq T^c = 1$ is then given by

$$u(x,t) = \begin{cases} 0, & x \leq 0; \\ 2(x/t), & 0 < x < x_s(t); \\ 0, & x \geq x_s(t). \end{cases} \tag{8}$$

A plot of the exact solution compared with numerical solution is shown in Fig. 1 at time $T = 1.0$ and $T = 1.5$ based on (3) and (8).

Problem 2.

(a) We refer to text note for details.

(b)

$$u_T = u_w|_{x=0} + u_o|_{x=0} = \frac{Q}{A} + 0 = \frac{Q}{A}.$$

$$F(S) = f(S)\Big[1 - \lambda_o g \sin(\alpha)\frac{\rho_w - \rho_o}{u_T}\Big]$$

(c) We find

$$S^* \approx 0.6 \qquad \Rightarrow \qquad V = f(S^*)/S^* \approx 0.88/0.6 = 1.46 \qquad \Rightarrow \qquad x^* \approx VT = 0.73$$

Moreover,

$$S = 0.7: \qquad f'(0.7) \approx 0.5 \qquad \Rightarrow \quad x_S = f'(S)T = 0.5 * 0.5 = 0.25$$

$$S = 0.8: \qquad f'(0.8) \approx 0.15 \qquad \Rightarrow \quad x_S = f'(S)T = 0.15 * 0.5 \approx 0.07$$

See Fig. 2 for a plot of solution.

(d) Mass conservation:

$$A_I = A_{II},$$

where

$$A_I = \int_0^1 f'(S)T\,dS, \qquad A_{II} = x^*S^* + \int_{S^*}^1 f'(S)T\,dS = f'(S^*)TS^* + T(f(1) - f(S^*))$$

This gives the relation

$$f'(S^*) = \frac{f(S^*)}{S^*}$$

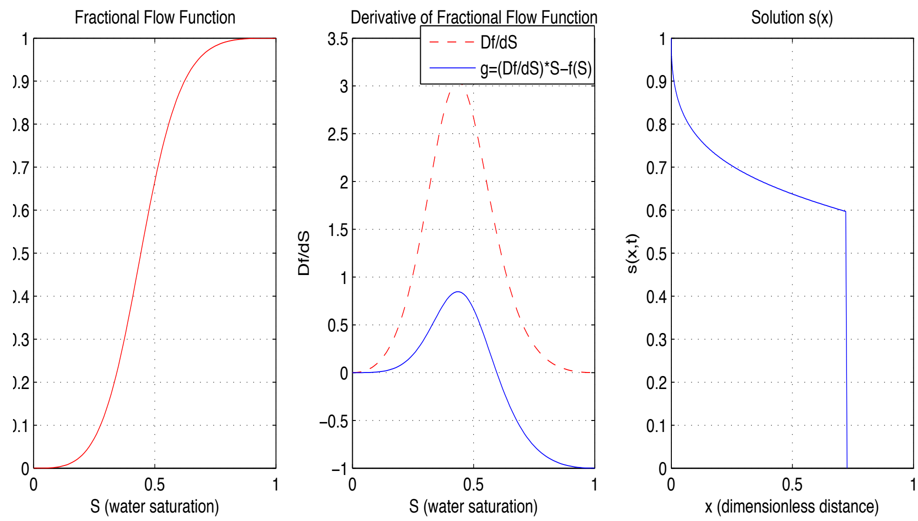

FIGURE 2. Solution of BL

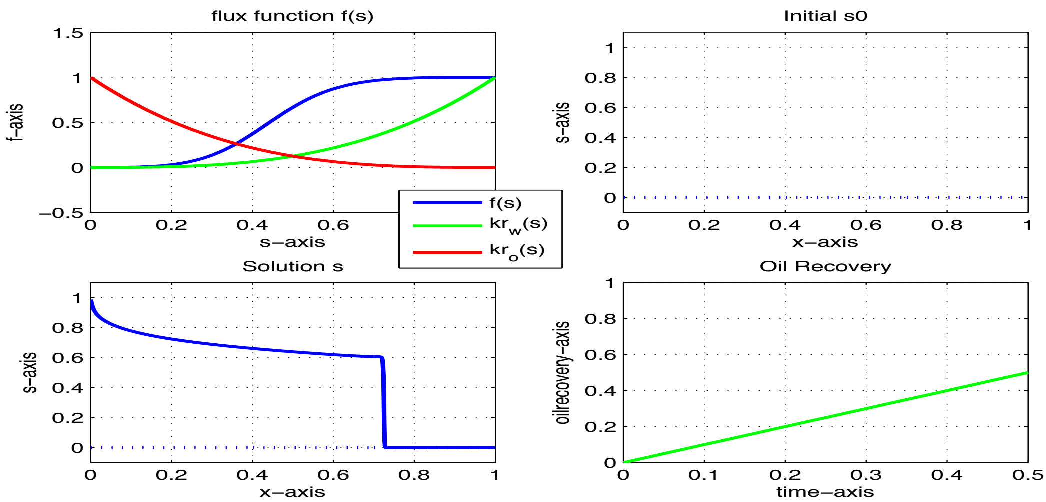

FIGURE 3. Horizontal

(e) See Fig. 3, 4, and 5 for a comparison of the three different cases.

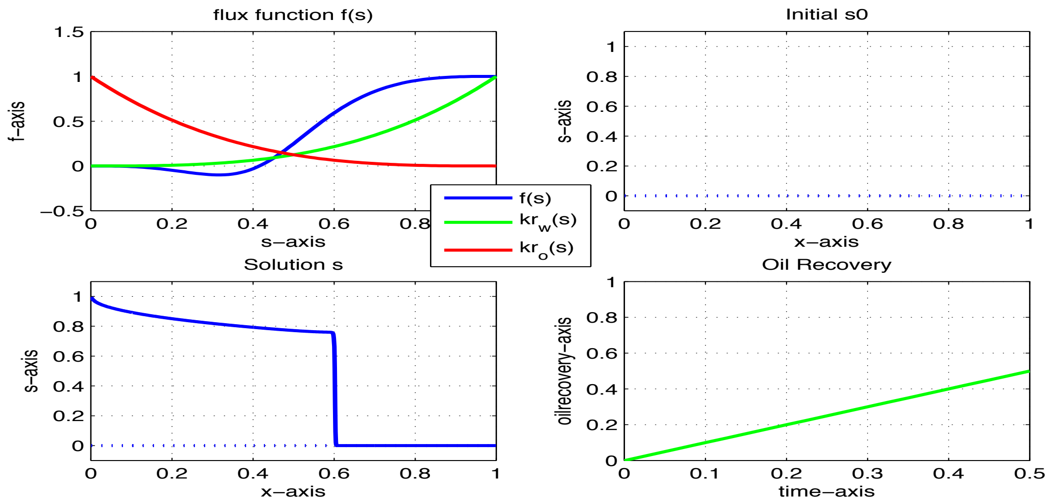

FIGURE 4. Upward dip

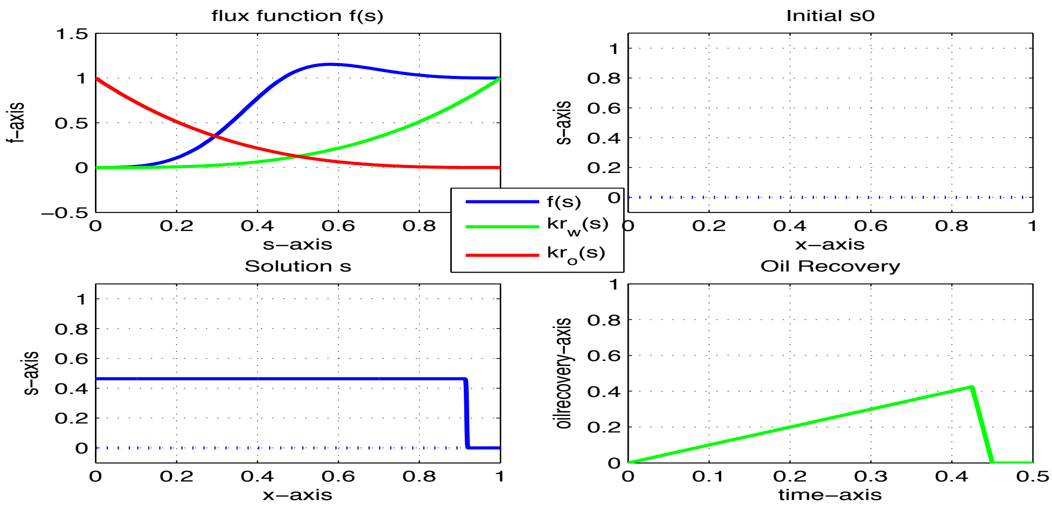

FIGURE 5. Downward dip