DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

EKSAMEN I: ÅMA290 Matematikk 3 - vektoranalyse

DATO: 14. desember 2009 kl. 0900 - 1300

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30.

Universitetet i Stavanger

OPPGAVESETTET BESTÅR AV 3 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = (3t-2)\,\mathbf{i} + 4t\,\mathbf{j} + \mathbf{k}$; $\quad 0 \le t \le 1$.

a) Beregn kurveintegralet

$$\int_C (xy + xz)\, ds.$$

b) Vis at vektorfeltet $\mathbf{F}(x,y,z) = (2x - 2xy)\mathbf{i} + (2yz - x^2)\,\mathbf{j} + y^2\,\mathbf{k}$ er konservativt.

c) Finn en potensialfunksjon (skalarfelt) til vektorfeltet $\mathbf{F}$.

d) Beregn kurveintegralet

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

OPPGAVE 2

La S være den delen av paraboloiden $z = 9 - x^2 - y^2$ som ligger over xy -planet, og la T være legemet avgrenset av flaten S og xy -planet.

a) Beregn integralet

$$\iiint_T 5\sqrt{z}\, dV. \qquad \text{(Hint: bruk sylinderkoordinater).}$$

b) Beregn flateintegralet

$$\iint_S \frac{10 - z}{\sqrt{4x^2 + 4y^2 + 1}}\, dS.$$

OPPGAVE 3

Gitt vektorfeltet $\mathbf{F}(x, y, z) = 2z\mathbf{i} + (8x^2 - 3y)\,\mathbf{j} + (3x + y)\,\mathbf{k}$.
La T være den delen av 1. oktant som ligger under planet $2x + 2y + z = 2$.

a) Finn $div\,\mathbf{F}$ og $curl\,\mathbf{F}$.

b) Bruk divergensteoremet til å beregne flateintegralet

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

der S er randen (overflaten) til legemet T, og $\mathbf{n}$ er enhetsnormalvektor til S. $\mathbf{n}$ peker utover (fra T).

c) La C være trekanten med hjørner i $(1, 0, 0)$, $(0, 1, 0)$ og $(0, 0, 2)$. Omløpsretning langs C (orientering) er mot urviser, sett ovenfra.
Beregn kurveintegralet

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\frac{d\mathbf{r}}{dt}|\,dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1dx + F_2dy + F_3dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})\,dt.$$

Flateintegral av en funksjon f over en flate $S : g(x, y, z) = K$ (K er en konstant):

$$\iint_S f\,dS = \iint_R f\,\frac{|\nabla g|}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Stokes' teorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (\nabla\times\mathbf{F})\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Divergensteoremet (Gauss' teorem):

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_T \nabla\cdot\mathbf{F}\,dV.$$

$$div\,\mathbf{F} = \nabla\cdot\mathbf{F},\quad curl\,\mathbf{F} = \nabla\times\mathbf{F},\quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z)$.
Kulekoordinater: $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

ÅMA 290 Matematikk 3 – vektoranalyse.

Eksamen 14. desember 2009

Løsning

Oppgave 1.

C: $\vec{r}(t) = (3t-2)\vec{i} + 4t\vec{j} + \vec{k}$

$0 \le t \le 1.$

a) $x = 3t-2, \quad y = 4t, \quad z = 1$

$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 4, \quad \frac{dz}{dt} = 0.$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = 3\vec{i} + 4\vec{j}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Beregn $\int_C (xy+xz)\,ds.$

$ds = \left|\frac{d\vec{r}}{dt}\right| dt = 5\,dt$

$$\int_C (xy+xz)\,ds = \int_0^1 [(3t-2)\cdot 4t + (3t-2)\cdot 1]\cdot 5\,dt =$$

$$= 5 \cdot \int_0^1 \left(12t^2 - 8t + 3t - 2\right) dt$$

$$= 5 \int [12t^2 - 5t - 2] dt = 5\left[\frac{12}{3}t^3 - \frac{5}{2}t^2 - 2t\right]_0^1$$

$$= 5\left[4 \cdot 1 - \frac{5}{2} \cdot 1 - 2 \cdot 1\right] = 5\left[4 - \frac{5}{2} - 2\right]$$

$$= 5\left[2 - \frac{5}{2}\right] = 5 \cdot \left[\frac{4-5}{2}\right] = 5 \cdot \left(-\frac{1}{2}\right) = \underline{\underline{-\frac{5}{2}}}$$

b) $\vec{F}(x, y, z) = (2x - 2xy)\vec{i} + (2yz - x^2)\vec{j} + y^2\vec{k}$.

Vis at $\vec{F}$ er konservativt.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - 2xy & 2yz - x^2 & y^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y} y^2 - \frac{\partial}{\partial z}(2yz - x^2)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial z}(2x - 2xy)\right.$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2yz - x^2) - \frac{\partial}{\partial y}(2x - 2xy)\right]$$

$$= \vec{i}[2y - 2y] - \vec{j}[0] + \vec{k}[-2x + 2x] = \vec{0}.$$

$\nabla \times \vec{F} = \vec{0} \implies \vec{F}$ konservativt felt.

c) Finn potensialfunksjon f ($\vec{F} = \nabla f$).

$$\frac{\partial f}{\partial x} = 2x - 2xy \qquad \mathrm{I}$$

$$\frac{\partial f}{\partial y} = 2yz - x^2 \qquad \mathrm{II}$$

$$\frac{\partial f}{\partial z} = y^2 \qquad \mathrm{III}$$

I. $\frac{\partial f}{\partial x} = 2x - 2xy \Rightarrow f = \int (2x - 2xy)\,dx$

$$= x^2 - x^2 y + C_1(y, z).$$

Deriver mhp. y:

$$\frac{\partial f}{\partial y} = -x^2 + \frac{\partial C_1}{\partial y} \underset{\mathrm{II}}{=} 2yz - x^2 \Rightarrow$$

$$\frac{\partial C_1}{\partial y} = 2yz \Rightarrow C_1 = \int 2yz\,dy$$

$$= y^2 z + C_2(z).$$

$$\Rightarrow f = x^2 - x^2 y + y^2 z + C_2$$

Deriver mhp. z: $\frac{\partial f}{\partial z} = y^2 + C_2' \underset{\mathrm{III}}{=} y^2 \Rightarrow$

$C_2' = 0 \Rightarrow C_2 = C$. Velg $C = 0$:

$$\underline{\underline{f = x^2 - x^2 y + y^2 z}}$$

d) Beregn $\int_C \vec{F} \cdot d\vec{r}$.

$C: \quad x = 3t - 2, \quad y = 4t, \quad z = 1$

$\vec{F}$ konservativ $\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_A^B df = f\Big|_A^B$

der A og B er koordinatene til hhv. start- og sluttpunkt på C.

$A: \quad t = 0: \quad x = -2, \quad y = 0, \quad z = 1$

$B: \quad t = 1: \quad x = 3 - 2 = 1, \quad y = 4, \quad z = 1.$

$$\int_C \vec{F} \cdot dr = f\Big|_A^B = x^2 - x^2 y + y^2 z \Big|_{(-2,0,1)}^{(1,4,1)}$$

$$= 1 - 1 \cdot 4 + 4^2 \cdot 1 - [(-2)^2 - (-2)^2 \cdot 0 + 0 \cdot 1]$$

$$= 1 - 4 + 16 - 4 = \underline{\underline{9}}$$

Oppgave 2.

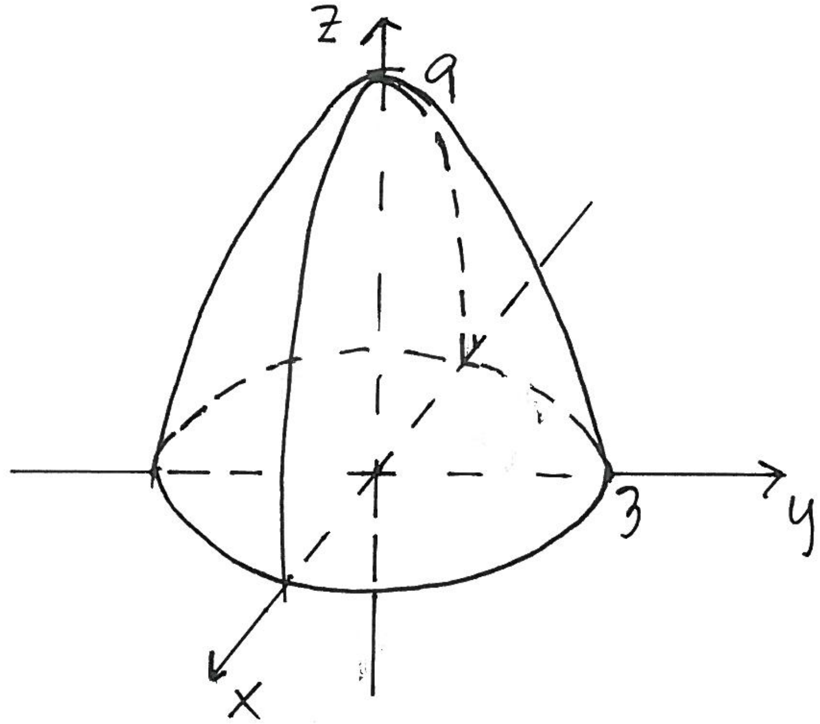
$$z = 9 - x^2 - y^2$$

Skisse : $z = 0 \Rightarrow x^2 + y^2 = 9$,

Sirkel med radius $= 3$.

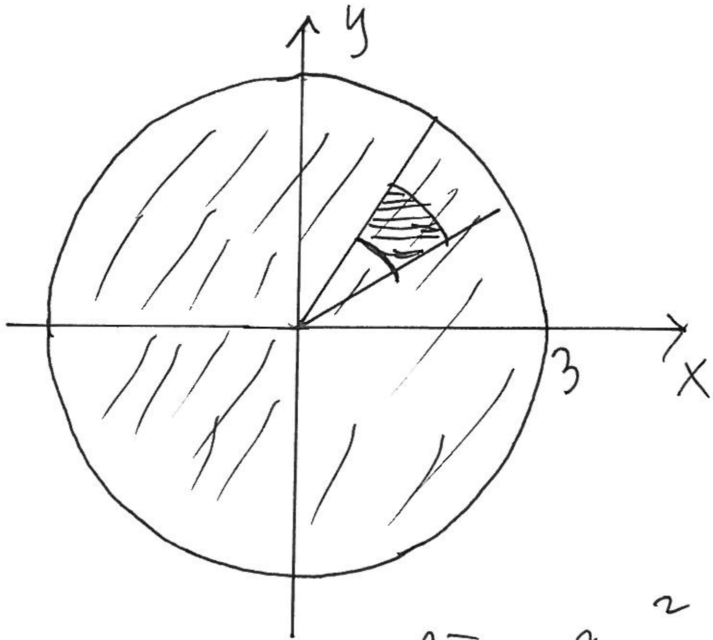
$x = 0$: $z = 9 - y^2$, parabel

$y = 0$: $z = 9 - x^2$, — " —



a) Beregn $\iiint_T 5\sqrt{z}\,dV$.

Sylinderkoordinater: $z = 9 - x^2 - y^2 = 9 - r^2$



$$\iiint_T 5\sqrt{z}\,dV = 5\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} z^{\frac{1}{2}}\,dz\,r\,dr\,d\theta$$

$$= 5\iint \frac{2}{3} z^{3/2}\Big|_0^{9-r^2} r\,dr\,d\theta$$

$$= 5\cdot\frac{2}{3}\iint (9-r^2)^{3/2}\, r\,dr\,d\theta$$

Subst.:

$u = 9-r^2$

$\frac{du}{dr} = -2r$

$dr = -\frac{du}{2r}$

$$= 5\cdot\frac{2}{3}\iint u^{3/2}\, r\left(-\frac{du}{2r}\right) d\theta$$

$$= -\frac{5}{3}\iint u^{3/2}\,du\,d\theta$$

$$= -\frac{5}{3}\cdot\frac{2}{5}\int u^{5/2}\,d\theta$$

$$= -\frac{2}{3}\int (9-r^2)^{5/2}\Big|_0^{3} d\theta = -\frac{2}{3}\left[0-9^{5/2}\right]\int_0^{2\pi} d\theta$$

$$= \frac{2}{3}\cdot 2\pi\cdot 9^{5/2} = \frac{4\pi}{3}\cdot 3^5 = 4\pi\cdot 3^4 = 4\pi\cdot 81 = \underline{\underline{324\pi}}$$

b) Beregn $\iint_S \frac{10-z}{\sqrt{4x^2+4y^2+1}}\,dS$

Definer $g(x,y,z) = x^2+y^2+z$

Paraboloiden er da nivåflaten $g=9$

$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k}$$

$$= 2x\vec{i} + 2y\vec{j} + \vec{k}$$

$$|\nabla g| = \sqrt{(2x)^2+(2y)^2+1} = \sqrt{4x^2+4y^2+1}$$

Projeksjonsplan : xy-planet. $\vec{P} = \vec{k}$.

$$dS = \frac{|\nabla g|}{|\nabla g \cdot \vec{P}|}\,dA = \frac{\sqrt{4x^2+4y^2+1}}{|\nabla g \cdot \vec{k}|}\,dA$$

$= \sqrt{4x^2+4y^2+1}\,dA$, der dA er er flateelement i xy-planet.

Projeksjon : $z=0 \Rightarrow x^2+y^2=9$, sirkel med radius $=3$.

$$\iint_S \frac{10-z}{\sqrt{4x^2+4y^2+1}}\,dS$$

$$\underset{\substack{\uparrow\\ \text{på } S.}}{=} \iint_R \frac{10-(9-x^2-y^2)}{\sqrt{4x^2+4y^2+1}}\sqrt{4x^2+4y^2+1}\,dA$$

$$= \iint_R (1+x^2+y^2)\,dA \underset{\substack{\uparrow\\ \text{polarkoord.}}}{=} \iint_R (1+r^2)\,dA$$

$$= \iint_R dA + \iint r^2\,dA.$$

$\iint_R dA$ = areal av sirkel med radius lik 3

$$= \pi\cdot 3^2 = 9\pi.$$

$$\iint r^2\,dA = \int_0^{2\pi}\int_0^3 r^2\, r\,dr\,d\theta = \iint r^3\,dr\,d\theta$$

$$= \frac{1}{4}\int r^4\Big|_0^3\,d\theta = \frac{1}{4}\cdot 3^4\int_0^{2\pi} d\theta = \frac{1}{4}\cdot 3^4\cdot 2\pi = \frac{81\pi}{2}$$

Totalt: $9\pi + \frac{81\pi}{2} = \frac{18+81}{2}\pi = \underline{\underline{\frac{99}{2}\pi}}$

Oppgave 3

$$\vec{F}(x,y,z) = 2z\vec{i} + (8x^2 - 3y)\vec{j} + (3x+y)\vec{k}$$

a) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2z) + \frac{\partial}{\partial y}(8x^2-3y) + \frac{\partial}{\partial z}(3x+y)$

$= 0 \quad -3 \quad +0 = \underline{\underline{-3}}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 8x^2-3y & 3x+y \end{vmatrix}$$

$= \vec{i}\left[\frac{\partial}{\partial y}(3x+y) - \frac{\partial}{\partial z}(8x^2-3y)\right]$

$- \vec{j}\left[\frac{\partial}{\partial x}(3x+y) - \frac{\partial}{\partial z}(2z)\right]$

$+ \vec{k}\left[\frac{\partial}{\partial x}(8x^2-3y) - \frac{\partial}{\partial y}(2z)\right]$

$= \vec{i}\cdot 1 - \vec{j}(3-2) + \vec{k}\cdot 16x = \underline{\underline{\vec{i} - \vec{j} + 16x\vec{k}}}.$

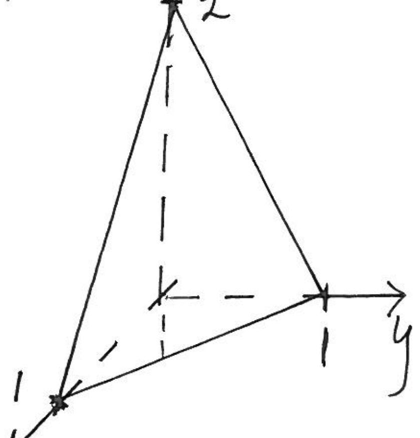
b) Planet: $2x + 2y + z = 2$.

Skisser planet i 1. oktant:

$x = y = 0 \Rightarrow z = 2$

$x = z = 0 \Rightarrow y = 1$

$y = z = 0 \Rightarrow x = 1$



Divergensteoremet:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_T \nabla \cdot \vec{F}\, dV = \iiint_T -3\, dV$$

$$= -3 \cdot V = -3 \cdot \frac{1}{3} A \cdot h = -1 \cdot \frac{1}{2} \cdot 2 = \underline{\underline{-1}}$$

(Alternativ: regn ut trippelintegralet)

c) Stokes' teorem: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS$

La S være den delen av planet $2x + 2y + z = 2$ i 1. oktant som skjærer koordinataksene i $(1,0,0)$, $(0,1,0)$ og $(0,0,2)$.

C er da randkurven til S.

Normalvektor til planet: $\vec{N} = [2,2,1]$.

(Alternativ: definer $g(x,y,z) = 2x + 2y + z$, $\vec{N} = \nabla g$).

enhetsnormalvektor:

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{[2,2,1]}{\sqrt{2^2+2^2+1}} = \frac{[2,2,1]}{\sqrt{9}} = \frac{[2,2,1]}{3}.$$

$$\nabla \times \vec{F} \cdot \vec{n} = [1,-1,16x] \cdot \frac{[2,2,1]}{3}$$

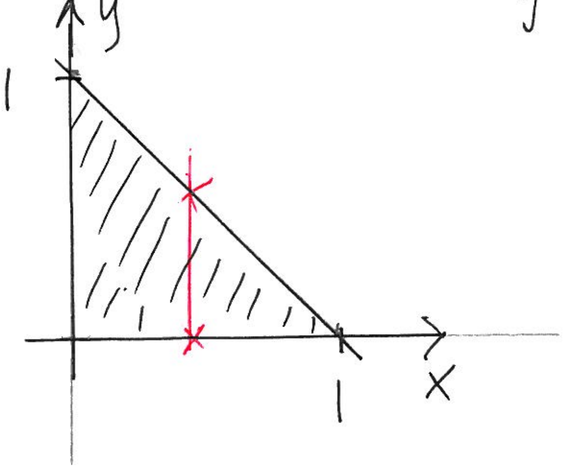
$$= \frac{2-2+16x}{3} = \frac{16x}{3}$$

$$dS = \frac{|\vec{N}|}{|\vec{N} \cdot \vec{k}|} dA = \frac{3}{1} dA = 3\,dA.$$

Projeksjonsplan: xy-planet. $\vec{P} = \vec{k}$.
dA er flateelement i xy-planet.

$$\Rightarrow \nabla \times \vec{F} \cdot \vec{n}\, dS = \frac{16x}{3} \cdot 3\,dA = 16x\,dA.$$

Projeksjon: $z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow$
$y = 1 - x.$



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \iint 16x\, dA$$

$$= 16 \int_0^1 \int_0^{1-x} x\, dy\, dx = 16 \int xy \Big|_0^{1-x} dx$$

$$= 16 \int x(1-x)\, dx = 16 \int (x - x^2)\, dx$$

$$= 16 \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 \right]_0^1 = 16 \left[\tfrac{1}{2} - \tfrac{1}{3} \right]$$

$$= 16 \left[\frac{3-2}{6} \right] = 16 \cdot \frac{1}{6} = \underline{\underline{\frac{8}{3}}}$$