DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

EKSAMEN I: ÅMA290 Matematikk 3 - vektoranalyse

DATO: 28. februar 2011 kl. 0900 - 1200

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30.

Universitetet i Stavanger

OPPGAVESETTET BESTÅR AV 3 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = (2t-2)\,\mathbf{i} - 2t\,\mathbf{j} + (t+1)\,\mathbf{k}$; $\quad 0 \le t \le 1$.

a) Finn enhetstangentvektor til C.

b) Beregn kurveintegralet

$$\int_C (y^2 + 2x)\, ds.$$

Gitt vektorfeltet $\mathbf{F}(x,y,z) = x\,\mathbf{i} + 3yz\,\mathbf{j} + y\,\mathbf{k}$.

c) Beregn kurveintegralet

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 2

Gitt vektorfeltet $\mathbf{F}(x,y,z) = (2x^2y - 3x)\mathbf{i} + (x^3 + z)\,\mathbf{j} + (y - x)\,\mathbf{k}$.

a) Finn $curl\,\mathbf{F}$.

b) La C være skjæringskurven mellom paraboloiden $z = 4 - x^2 - y^2$ og $xy-$ planet. Omløpsretning langs C (orientering) er mot urviser, sett ovenfra. Beregn kurveintegralet

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 3

La T være den delen av 1. oktant som ligger under planet $4x + y + z = 4$.

a) Beregn integralet

$$\iiint_T x\,dV.$$

Gitt vektorfeltet $\mathbf{F}(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$.
La S være den delen av planet $4x + y + z = 4$ som ligger i 1. oktant.

b) Beregne flateintegralet

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS, \qquad \text{(fluksen av } \mathbf{F}\text{)}$$

der $\mathbf{n}$ er enhetsnormalvektor til S og peker utover fra T.

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $\quad a \leq t \leq b$:

$$\int\limits_C f\,ds = \int_a^b f(\mathbf{r}(t))|\frac{d\mathbf{r}}{dt}|\,dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int\limits_C \mathbf{F}\cdot\mathbf{T}\,ds = \int\limits_C \mathbf{F}\cdot d\mathbf{r} = \int\limits_C F_1dx + F_2dy + F_3dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})\,dt.$$

Flateintegral av en funksjon f over en flate $S : g(x,y,z) = K$ (K er en konstant):

$$\iint\limits_S f\,dS = \iint\limits_R f\,\frac{|\nabla g|}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Stokes' teorem:

$$\oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint\limits_R (\nabla\times\mathbf{F})\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Divergensteoremet (Gauss' teorem):

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_T \nabla\cdot\mathbf{F}\,dV.$$

$$div\,\mathbf{F} = \nabla\cdot\mathbf{F}, \quad curl\,\mathbf{F} = \nabla\times\mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z)$.
Kulekoordinater: $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

ÅMA 290 Matematikk 3 - vektoranalyse

Eksamen 28. februar 2011.

Løsning.

Oppgave 1

$C: \quad \vec{r}(t) = (2t-2)\vec{i} - 2t\vec{j} + (t+1)\vec{k}.$

$0 \leq t \leq 1.$

Dvs. : $x = 2t-2$, $y = -2t$, $z = t+1$.

a) Enhetstangentvektor $\vec{T} = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = -2, \quad \frac{dz}{dt} = 1.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{2^2 + (-2)^2 + 1} = \sqrt{9} = 3$$

$$\underline{\underline{\vec{T} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3}}}$$

b) $ds = \left|\frac{d\vec{r}}{dt}\right| dt = 3\,dt.$

$$\int_C (y^2 + 2x)\,ds = \int_0^1 (y^2 + 2x)\cdot 3\,dt$$

$$= 3\int [(-2t)^2 + 2(2t-2)]\,dt$$

$$= 3\int (4t^2 + 4t - 4)\,dt = 3\cdot 4\int_0^1 (t^2 + t - 1)\,dt$$

$$= 12\left[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - t\right]_0^1 = 12\left[\tfrac{1}{3} + \tfrac{1}{2} - 1\right]$$

$$= 4 + 6 - 12 = \underline{\underline{-2}}$$

c) $\vec{F} = x\vec{i} + 3yz\vec{j} + y\vec{k}.$

$$\int_C \vec{F}\cdot d\vec{r} = \int \left(\vec{F}\cdot \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^1 \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int \left(x\frac{dx}{dt} + 3yz\frac{dy}{dt} + y\frac{dz}{dt}\right) dt$$

$$= \int [(2t-2)\cdot 2 + 3(-2t)(t+1)\cdot(-2) - 2t]\,dt$$

$$= \int [4t - 4 + 12t^2 + 12t - 2t]\,dt$$

$$= \int_0^1 \left[12t^2 + 14t - 4\right] dt$$

$$= 12 \cdot \frac{1}{3}t^3 + 14 \cdot \frac{1}{2}t - 4t \Big|_0^1 = 4t^3 + 7t - 4t \Big|_0^1$$

$$= 4 + 7 - 4 \underline{\underline{= 7}}$$

Oppgave 2

$$\vec{F}(x, y, z) = (2x^2y - 3x)\vec{i} + (x^3 + z)\vec{j} + (y - x)\vec{k}$$

a)

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2y - 3x & x^3 + z & y - x \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y - x) - \frac{\partial}{\partial z}(x^3 + z)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(y - x) - \frac{\partial}{\partial z}(2x^2y - 3x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(x^3 + z) - \frac{\partial}{\partial y}(2x^2y - 3x)\right]$$

$$= \vec{i}\left[1 - 1\right] - \vec{j}\left[-1\right] + \vec{k}\left[3x^2 - 2x^2\right]$$

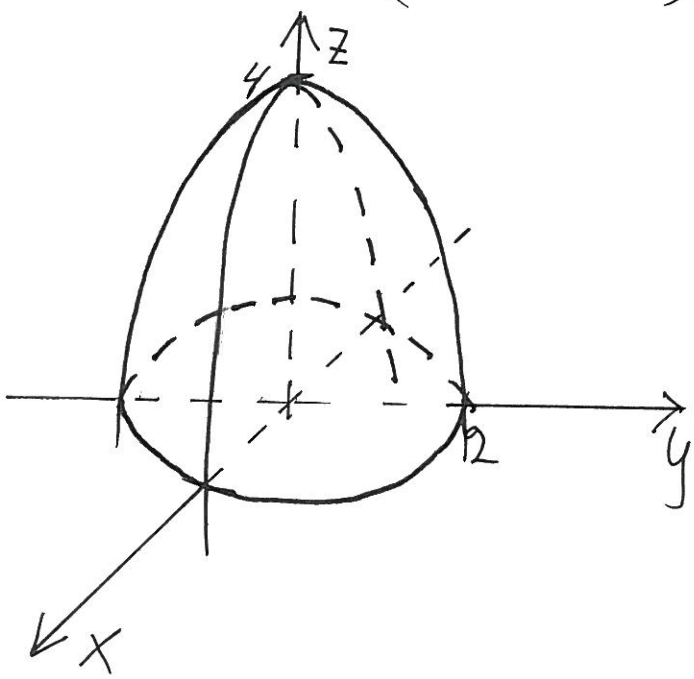
$$= \underline{\underline{\vec{j} + x^2\vec{k}}}$$

b) Paraboloiden $z = 4 - x^2 - y^2$

skisser: $z=0$: $x^2+y^2=4$ (sirkel)

$x=0$: $z=4-y^2$, $y=0$: $z=4-x^2$

(parabler).



Skjæring mellom paraboloiden og xy-planet:

$z=0$: $x^2+y^2=4$, sirkel med radius = 2

Beregn $\oint_C \vec{F}\cdot d\vec{r}$. Bruk Stokes' teorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS$$

La S være sirkelskiva $S_1 : x^2+y^2 \leq 4$ i xy-planet

S_1 har C som randkurve.

$\vec{n} = \vec{k}$. $dS = dA$ = flateelement i xy-planet.

Bruk polarkoordinater:

Stokes' teorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \nabla \times \vec{F} \cdot \vec{k} \, dA$$

$$= \int_0^{2\pi}\int_0^2 (\vec{j} + x^2\vec{k}) \cdot \vec{k} \, r\,dr\,d\theta = \iint x^2 \, r\,dr\,d\theta$$

$$= \iint (r\cos\theta)^2 \, r\,dr\,d\theta = \iint r^3 \cos^2\theta \, dr\,d\theta$$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$	$= \iint r^3 \cdot \frac{1}{2}[\cos(2\theta)+1]\,dr\,d\theta$
$= \cos^2\theta - (1-\cos^2\theta)$	
$= 2\cos^2\theta - 1 \Rightarrow$	$= \frac{1}{2}\cdot\frac{1}{4}\int r^4 \Big|_0^2 [\cos(2\theta)+1]\,d\theta$
$2\cos^2\theta = \cos(2\theta)+1$	
$\Rightarrow \cos^2\theta = \frac{1}{2}[\cos(2\theta)+1]$	$= \frac{1}{8}\cdot 2^4 \int [\cos(2\theta)+1]\,d\theta$

$$= 2 \cdot \Big[\underbrace{\tfrac{1}{2}\sin(2\theta)}_{=0} + \theta\Big]\Big|_0^{2\pi}$$

$$= 2 \cdot [0 + 2\pi] = \underline{\underline{4\pi}}$$

Oppgave 3

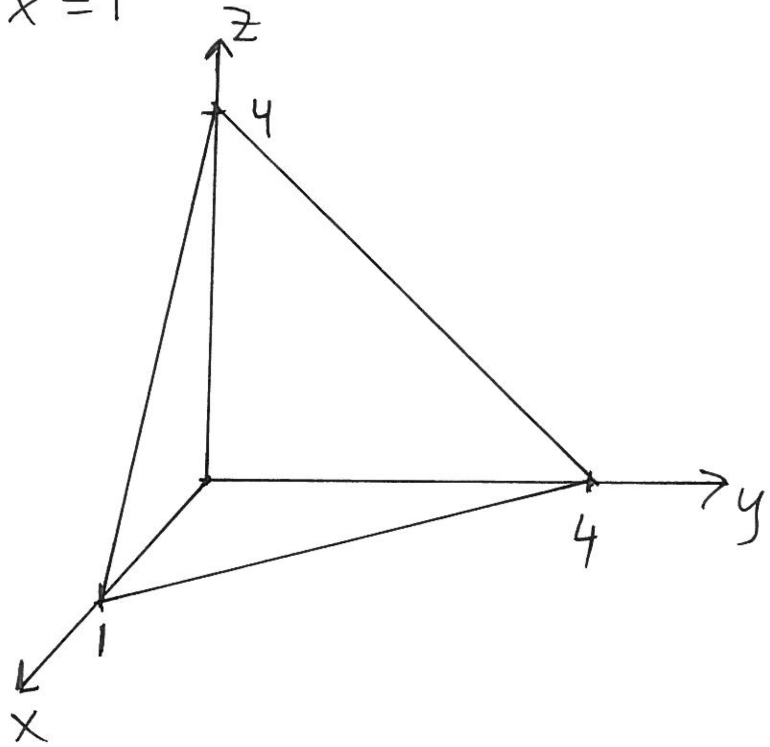
Planet $4x + y + z = 4$.

Skisser planet:

$x = y = 0 \Rightarrow z = 4$

$x = z = 0 \Rightarrow y = 4$

$y = z = 0 \Rightarrow x = 1$



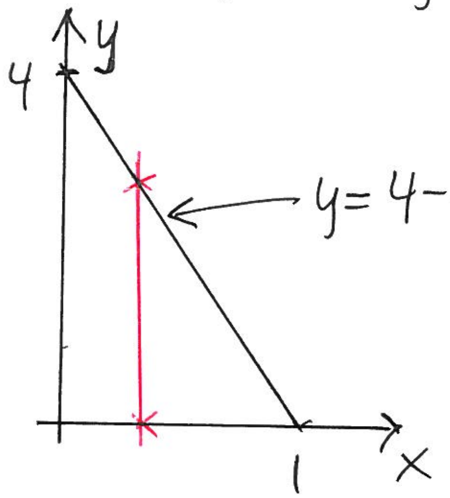
a) $\iiint_T x\,dV.$

Integrasjon i z-retning først.

$z = 4 - 4x - y$

Projeksjon av planet i xy-planet:

$z = 0$: $4 - 4x - y = 0 \Rightarrow y = 4 - 4x.$



$$\iiint_T x\,dV = \int_0^1 \int_0^{4-4x} \int_0^{4-4x-y} x\,dz\,dy\,dx$$

$$= \iint x z \Big|_0^{4-4x-y} dy\,dx$$

$$= \iint x(4-4x-y)\,dy\,dx = \iint (4x - 4x^2 - xy)\,dy\,dx$$

$$= \int_0^1 4xy - 4x^2y - \tfrac{1}{2}xy^2 \Big|_0^{4-4x} dx$$

$$= \int y\left[4x - 4x^2 - \tfrac{1}{2}xy\right]\Big|_0^{4-4x} dx$$

$$= \int (4-4x)\left[4x - 4x^2 - \tfrac{1}{2}x(4-4x)\right] dx$$

$$= 4\int (1-x)\left[4x - 4x^2 - 2x + 2x^2\right] dx$$

$$= 4\int (1-x)\left[2x - 2x^2\right] = 8\int (1-x)\left[x - x^2\right] dx$$

$$= 8\int (x - x^2 - x^2 + x^3) dx = 8\int (x - 2x^2 + x^3) dx$$

$$= 8\left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4\right]\Big|_0^1$$

$$= 8\left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right] = 4 - \tfrac{16}{3} + 2 = 6 - \tfrac{16}{3}$$

$$= \frac{18-16}{3} = \underline{\underline{\frac{2}{3}}}$$

b) $\vec{F} = \vec{i} - \vec{j} - \vec{k}$.

Normalen til planet $4x + y + z = 4$:

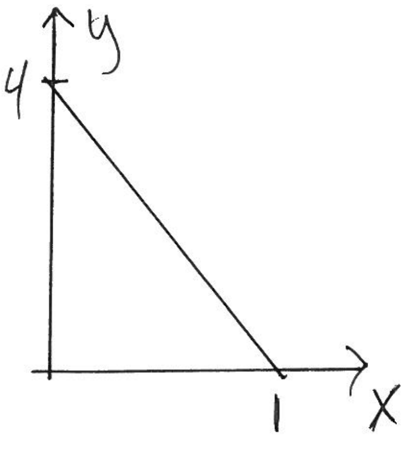
$\vec{N} = [4, 1, 1]$.

Enhetsnormalvektor $\vec{n} = \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{[4,1,1]}{|\vec{N}|}$

Projeksjonsplan: xy-planet $\Rightarrow \vec{P} = \vec{k}$:

$$dS = \frac{|\vec{N}|}{|\vec{N} \cdot \vec{P}|} dA = \frac{|\vec{N}|}{|\vec{N} \cdot \vec{k}|} dA = |\vec{N}| dA,$$

der dA er flateelement i xy-planet.



$$\vec{F} \cdot \vec{n} dS = [1, -1, 1] \cdot \frac{[4,1,1]}{|\vec{N}|} |\vec{N}| dA$$

$$= (4 - 1 - 1) dA = 2 dA.$$

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint 2 dA$$

$= 2 \cdot$ areal av planets projeksjon i xy-planet

$= 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = \underline{\underline{4}}$