DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

EKSAMEN I: ÅMA290 Matematikk 3 - vektoranalyse

DATO: 13. desember 2011 kl. 1300 - 1600

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30, Citizen SR-270X.

Universitetet i Stavanger

OPPGAVESETTET BESTÅR AV 3 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = -2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4t\,\mathbf{k};\quad 0 \leq t \leq \pi.$

a) Finn enhetstangentvektor til C i punktet svarende til $t = \pi/2$.

b) Beregn kurveintegralet

$$\int_C (-xy + z)\,ds.$$

Gitt vektorfeltet $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\,\mathbf{j} + 2\,\mathbf{k}$.

c) Beregn kurveintegralet

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 2

Gitt vektorfeltet $\mathbf{F}(x, y, z) = 2xz\mathbf{i} + xy\,\mathbf{j} - xz\,\mathbf{k}$.
La T være legemet avgrenset av øvre halvkuleflate $x^2 + y^2 + z^2 = 1 \;\; (z \geq 0)$ og xy -planet.

a) Finn $div\,\mathbf{F}$ og $curl\,\mathbf{F}$.

b) (i) Bruk divergensteoremet til å beregne flateintegralet (fluksen)

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,dS,$$

der S er randen (overflaten) til legemet T, og $\mathbf{n}$ er enhetsnormalvektor til S. $\mathbf{n}$ peker utover (fra T).
(ii) Finn fluksen av vektorfeltet $\mathbf{F}(x, y, z)$ gjennom sirkelskiva $x^2 + y^2 \leq 1$ i xy -planet, i retning ut fra legemet T.

OPPGAVE 3

La S være den delen av paraboloiden $z = 2x^2 + 2y^2$ som ligger under planet $z = 6$. Beregn flateintegralet

$$\iint_S \frac{1}{8z+1}\, dS.$$

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\frac{d\mathbf{r}}{dt}|\,dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1dx + F_2dy + F_3dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})\,dt.$$

Flateintegral av en funksjon f over en flate $S: g(x,y,z) = K$ (K er en konstant):

$$\iint_S f\,dS = \iint_R f\,\frac{|\nabla g|}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Stokes' teorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_R (\nabla\times\mathbf{F})\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Divergensteoremet (Gauss' teorem):

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_T \nabla\cdot\mathbf{F}\,dV.$$

$$div\,\mathbf{F} = \nabla\cdot\mathbf{F},\quad curl\,\mathbf{F} = \nabla\times\mathbf{F},\quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z)$.
Kulekoordinater: $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

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Eksamen 13. desember 2011

Løsning

Oppgave 1.

Kurven C : $\vec{r}(t) = -2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 4t\,\vec{k}$
$0 \le t \le \pi$

Dvs. : $x = -2\cos t$, $y = 2\sin t$, $z = 4t$.

a) Enhetstangentvektor $\vec{T} = \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$\frac{dx}{dt} = 2\sin t$, $\frac{dy}{dt} = 2\cos t$, $\frac{dz}{dt} = 4$.

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$= 2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 4\vec{k}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 16}$$

$$= \sqrt{4(\cos^2 t + \sin^2 t) + 16} = \sqrt{20} = \sqrt{4\cdot 5} = \underline{2\sqrt{5}}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{2}} = 2\sin\left(\tfrac{\pi}{2}\right)\vec{i} + 2\cos\left(\tfrac{\pi}{2}\right)\vec{j} + 4\vec{k}$$

$$= 2\vec{i} + 4\vec{k}$$

$$\vec{T} = \frac{2\vec{i}+4\vec{k}}{2\sqrt{5}} = \underline{\underline{\frac{\vec{i}+2\vec{k}}{\sqrt{5}}}} \quad \left(= \frac{\sqrt{5}}{5}\left(\vec{i}+2\vec{k}\right)\right).$$

b) $ds = \left|\frac{d\vec{r}}{dt}\right| dt = 2\sqrt{5}\, dt$.

$$\int_C (-xy+z)\,ds = \int_C (-xy+z)\cdot 2\sqrt{5}\,dt$$

$$= 2\sqrt{5}\int_0^{\pi}\left(2\cos t\cdot 2\sin t + 4t\right)dt$$

$$= 8\sqrt{5}\int u\cos t\,\frac{du}{\cos t} + 8\sqrt{5}\int_0^{\pi} t\,dt$$

Subst.:
$u = \sin t$
$\frac{du}{dt} = \cos t$
$dt = \frac{du}{\cos t}$

$$= 8\sqrt{5}\int u\,du + 8\sqrt{5}\cdot\frac{1}{2}t^2\Big|_0^{\pi}$$

$$= 8\sqrt{5}\cdot\frac{1}{2}u^2 + 4\sqrt{5}\cdot\pi^2$$

$$= \underbrace{4\sqrt{5}\,\sin^2 t\Big|_0^{\pi}}_{=0} + 4\sqrt{5}\pi^2 = \underline{\underline{4\sqrt{5}\pi^2}}$$

c) $\vec{F} = -y\vec{i} + x\vec{j} + 2\vec{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int \left(-y \frac{dx}{dt} + x \frac{dy}{dt} + 2\frac{dz}{dt}\right) dt$$

$$= \int \left(-2\sin t \cdot 2\sin t - 2\cos t \cdot 2\cos t + 2 \cdot 4\right) dt$$

$$= \int_0^{\pi} \left[-4(\sin^2 t + \cos^2 t) + 8\right] dt = \int (-4+8)\, dt$$

$$= 4\int_0^{\pi} dt = 4t \Big|_0^{\pi} = \underline{\underline{4\pi}}$$

Oppgave 2

$\vec{F}(x,y,z) = 2xz\vec{i} + xy\vec{j} - xz\vec{k}$.

a) $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(-xz)$

$$= 2z + x - x = \underline{\underline{2z}}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & xy & -xz \end{vmatrix} =$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(xy)\right]$$

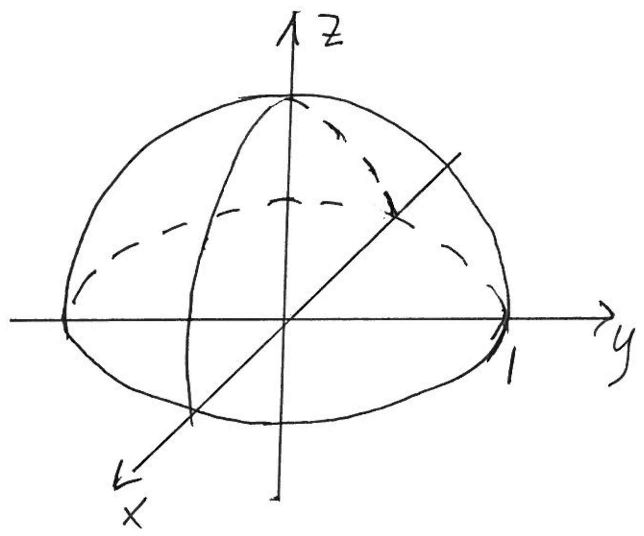
$$-\vec{j}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(2xz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(2xz)\right]$$

$$= \vec{i}\,[0] - \vec{j}\,[-z - 2x] + \vec{k}\,[y]$$

$$= \underline{\underline{(2x+z)\vec{j} + y\vec{k}}}\,.$$

b)



(i) Divergensteoremet :

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_T \nabla\cdot\vec{F}\,dV = \iiint_T 2z\,dV =$$

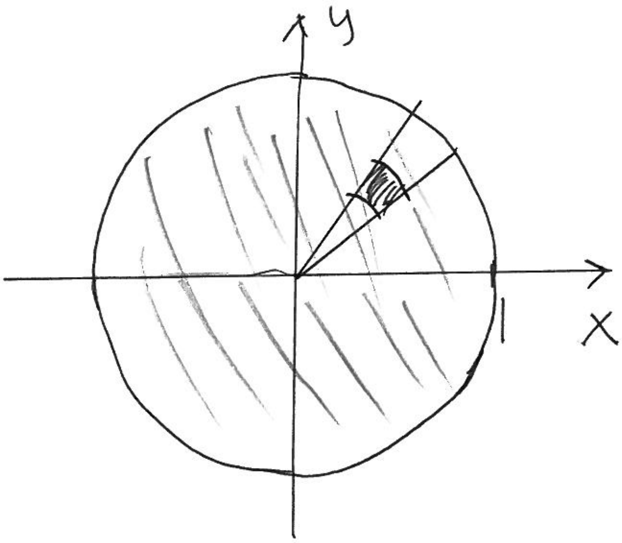
Kula :

$x^2+y^2+z^2=1$

$\Rightarrow z = \sqrt{1-(x^2+y^2)}$

$= \sqrt{1-r^2}$ i sylinderkoord

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} 2z \quad r\,dz\,dr\,d\theta = 2\iint \frac{1}{2}z^2\Big|_0^{\sqrt{1-r^2}} r\,dr\,d\theta$$



$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta =$$

$$\iint (r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{4}r^4\Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{4}\right)d\theta = \frac{1}{4}\int d\theta = \frac{1}{4}\cdot 2\pi = \underline{\underline{\frac{\pi}{2}}}$$

(ii) S: Sirkelflata $x^2+y^2 \le 1$ i xy-planet.

$$\vec{F} = 2xz\vec{i} + xy\vec{j} - xz\vec{k}$$

Enhetsnormalvektor til S: $\vec{n} = -\vec{k}$.

$$\vec{F}\cdot\vec{n} = xz = 0$$

$z=0$ i xy-planet

$$\Rightarrow \text{Fluks} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint xz\,dS = \underline{\underline{0}}$$

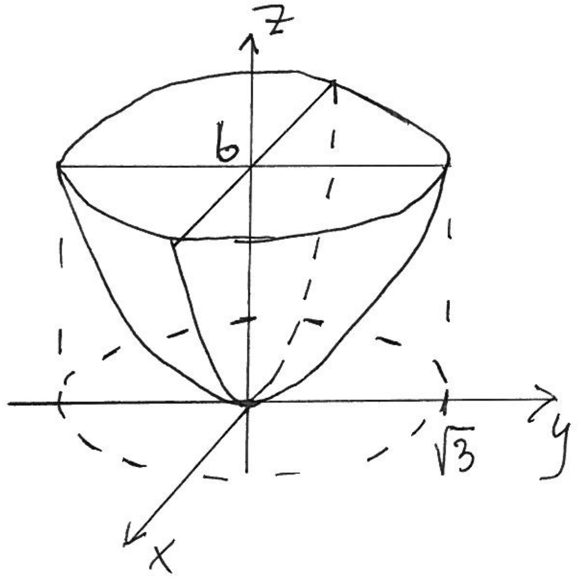
Oppgave 3

$S: \; z = 2x^2 + 2y^2$.

Skjæring mellom paraboloiden og $z = 6$:

$$6 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 3.$$

Sirkel med radius $= \sqrt{3}$.



Beregn $\iint_S \frac{1}{8z+1} \, dS$

dS: flateelement på S

Definer $g(x, y, z) = 2x^2 + 2y^2 - z \qquad (= 0)$.

S er nivåflaten $g = 0$.

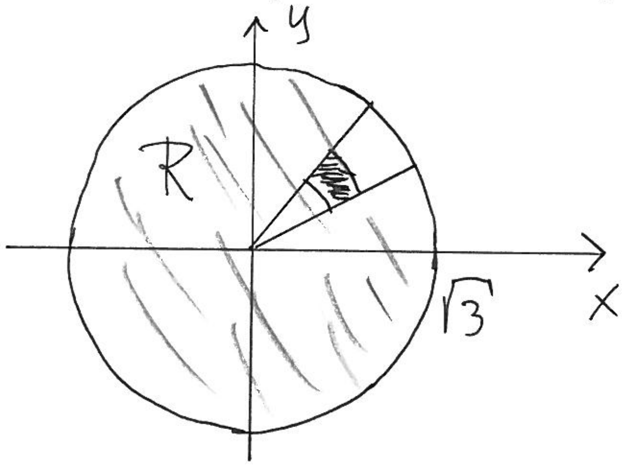
$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k} = 4x\vec{i} + 4y\vec{j} - \vec{k}$$

$$|\nabla g| = \sqrt{(4x)^2 + (4y)^2 + 1} = \sqrt{16x^2 + 16y^2 + 1}$$

$$dS = \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} \, dA \, ,$$ der $\vec{p}$ er

enhetsnormalvektor til projeksjonsplanet.
dA er flateelement i projeksjonsplan.

Projeksjonsplan: xy-planet.



$$\vec{p} = \vec{k}$$

$$|\nabla g \cdot \vec{k}| = |-1| = 1.$$

$$dS = \frac{|\nabla g|}{|\nabla g \cdot \vec{k}|} \, dA = \sqrt{16(x^2+y^2)+1} \cdot r\,dr\,d\theta.$$

(polarkoord.)

$$\iint_S \frac{1}{8z+1} \, dS \underset{\text{på } S}{=} \iint_R \frac{1}{8(2x^2+2y^2)+1} \sqrt{16(x^2+y^2)+1} \; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{\sqrt{16r^2+1}}{16r^2+1} \, r\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{1}{\sqrt{16r^2+1}} \, r\,dr\,d\theta$$

Subst.:

$u = 16r^2 + 1$

$\frac{du}{dr} = 32r$

$r\,dr = \frac{du}{32}$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{1}{\sqrt{u}} \frac{du}{32}\, d\theta$$

$$= \frac{1}{32} \int\int u^{-\frac{1}{2}}\, du\, d\theta$$

$$= \frac{1}{32} \cdot 2 \int_0^{2\pi} u^{1/2}\, d\theta$$

$$= \frac{1}{16} \int_0^{2\pi} (16r^2+1)^{1/2} \Big|_0^{\sqrt{3}}\, d\theta$$

$$= \frac{1}{16} \int \left((16 \cdot 3 + 1)^{1/2} - 1 \right) d\theta = \frac{1}{16} \int \left(\sqrt{49} - 1 \right) d\theta$$

$$= \frac{6}{16} \int_0^{2\pi} d\theta = \frac{3}{8} \cdot 2\pi = \underline{\underline{\frac{3\pi}{4}}}$$