Universitetet
i Stavanger

DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

EKSAMEN I: MAT300 Vektoranalyse

DATO: 26. februar 2014 kl. 0900 - 1300

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30, Citizen SR-270X.

OPPGAVESETTET BESTÅR AV 3 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = (-2+2t)\,\mathbf{i} + (1-2t)\,\mathbf{j} \ + 2t\,\mathbf{k}; \quad 0 \leq t \leq 1.$

a) Finn enhetstangentvektor til C.

b) Beregn kurveintegralet

$$\int_C (2x + y + 3z^2)\, ds.$$

Gitt vektorfeltet $\mathbf{F}(x,y,z) = (x+y)\mathbf{i} + 2z^2\,\mathbf{j} + y^2\,\mathbf{k}$.

c) Beregn kurveintegralet

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 2

Gitt vektorfeltet $\mathbf{F}(x,y,z) = (xz + x^3)\mathbf{i} + (x^3 z + y^3)\,\mathbf{j} + z\,\mathbf{k}$.
La T være legemet avgrenset av sylinderen $x^2 + y^2 = 4$, og planene $z = -2$ og $z = 2$.

a) Finn $div\,\mathbf{F}$ og $curl\,\mathbf{F}$.

b) Bruk divergensteoremet til å beregne flateintegralet

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

der S er randen (overflaten) til legemet T. $\hat{\mathbf{N}}$ er enhetsnormalvektor til S, og peker utover (fra T).

c) La C være skjæringskurven mellom sylinderen $x^2 + y^2 = 4$, og planet $z = 2$. Omløpsretning langs C (orientering) er mot urviser, sett ovenfra.
Beregn kurveintegralet

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 3

La området T være den delen av 1. oktant som ligger under planet $4x + y + z = 4$. La S være den delen av planet $4x + y + z = 4$ som ligger i 1. oktant.

a) (i) Finn enhetsnormalvektor til planet S. Normalvektoren har retning ut fra T.
(ii) Beregn fluks

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

av vektorfeltet $\mathbf{F}(x, y, z) = yz\mathbf{i} - 3yz\,\mathbf{j} + (x - yz)\,\mathbf{k}$.
$\hat{\mathbf{N}}$ er enhetsnormalvektor til S, og peker utover (fra T).

b) Beregn

$$\iiint_T y\, dV.$$

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int\limits_C f\, ds = \int_a^b f(\mathbf{r}(t))|\frac{d\mathbf{r}}{dt}|\, dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int\limits_C \mathbf{F}\cdot\hat{\mathbf{T}}\, ds = \int\limits_C \mathbf{F}\cdot d\mathbf{r} = \int\limits_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})dt.$$

Flateintegral av en funksjon f over en flate $S : g(x, y, z) = K$ (**K** er en konstant):

$$\iint\limits_S f\, dS = \iint\limits_R f\, \frac{|\nabla g|}{|\nabla g\cdot\mathbf{P}|}\, dA.$$

Stokes' teorem:

$$\oint\limits_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{N}}\, dS = \iint\limits_R (\nabla\times\mathbf{F})\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{P}|}\, dA.$$

Divergensteoremet:

$$\iiint\limits_D \nabla\cdot\mathbf{F}\, dV = \iint\limits_S \mathbf{F}\cdot\hat{\mathbf{N}}\, dS.$$

$$div\,\mathbf{F} = \nabla\cdot\mathbf{F}, \quad curl\,\mathbf{F} = \nabla\times\mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z)$.
Kulekoordinater: $(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

MAT300 vektoranalyse

Eksamen 26. februar 2014.

Lösning.

Oppgave 1.

Kurven C : $\vec{r}(t) = (-2+2t)\vec{i} + (1-2t)\vec{j} + 2t\vec{k}$

$0 \leq t \leq 1$

Dvs : $x = -2+2t \Rightarrow \frac{dx}{dt} = 2$

$y = 1-2t \Rightarrow \frac{dy}{dt} = -2$

$z = 2t \Rightarrow \frac{dz}{dt} = 2$.

a) Enhetstangentvektor :

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} = \sqrt{2^2+(-2)^2+2^2}$$

$$= \sqrt{12} = \sqrt{4\cdot 3} = \underline{2\sqrt{3}}$$

$$\vec{T} = \frac{2\vec{i}-2\vec{j}+2\vec{k}}{2\sqrt{3}} = \underline{\underline{\frac{\vec{i}-\vec{j}+\vec{k}}{\sqrt{3}}}}$$

b) $ds = \left|\frac{d\vec{r}}{dt}\right| dt = 2\sqrt{3}\, dt$

$$\int_C (2x+y+3z^2)\, ds = 2\sqrt{3}\int_0^1 (2x+y+3z^2)\, dt$$

$$= 2\sqrt{3}\int [2\cdot(-2+2t) + 1-2t + 3(2t)^2]\, dt$$

$$= 2\sqrt{3}\int [-4+4t+1-2t+12t^2]\, dt$$

$$= 2\sqrt{3}\int [12t^2+2t-3]\, dt$$

$$= 2\sqrt{3}\left[\frac{1}{3}\cdot 12t^3 + t^2 - 3t\right]_0^1$$

$$= 2\sqrt{3}[4+1-3] = 2\sqrt{3}\cdot 2 = \underline{\underline{4\sqrt{3}}}$$

c) $\vec{F} = (x+y)\vec{i} + 2z^2\vec{j} + y^2\vec{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^1 \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

$$\int [(x+y)\cdot 2 + 2z^2\cdot(-2) + y^2\cdot 2]\, dt$$

$$= \int [(-2+2t+1-2t)\cdot 2 + 2(2t)^2\cdot(-2) + (1-2t)^2\cdot 2]\, dt$$

$$= \int [-1\cdot 2 + 8t^2\cdot(-2) + (1-4t+4t^2)\cdot 2]\, dt$$

$$= \int [-2 - 16t^2 + 2 - 8t + 8t^2]\,dt$$

$$= \int [-8t^2 - 8t]\,dt = -8\int (t^2 + t)\,dt$$

$$= -8\left[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2\right]_0^1 = -8\left[\tfrac{1}{3} + \tfrac{1}{2}\right]$$

$$= -8\left[\frac{2+3}{6}\right] = -8 \cdot \frac{5}{6} = \underline{\underline{-\frac{20}{3}}}$$

Opgave 2.

$$\vec{F}(x, y, z) = (xz + x^3)\vec{i} + (x^3 z + y^3)\vec{j} + z\vec{k}$$

a) $$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz + x^3) + \frac{\partial}{\partial y}(x^3 z + y^3) + \frac{\partial}{\partial z} z$$

$$= z + 3x^2 + 3y^2 + 1 = \underline{\underline{3(x^2 + y^2) + z + 1.}}$$

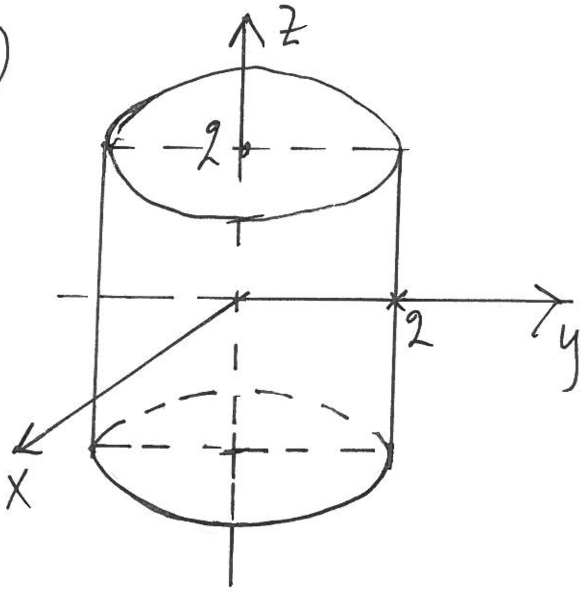
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz + x^3 & x^3 z + y^3 & z \end{vmatrix}$$

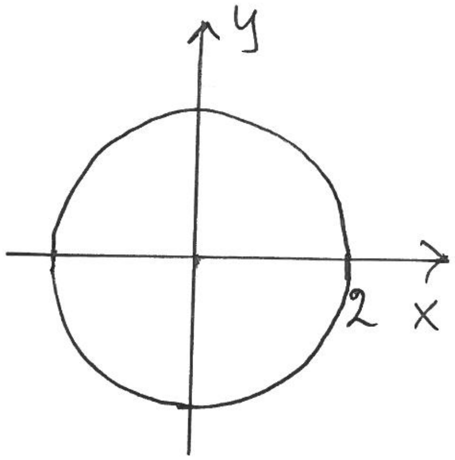
$$= \vec{i}\left[\frac{\partial}{\partial y} z - \frac{\partial}{\partial z}(x^3 z + y^3)\right] - \vec{j}\left[\frac{\partial}{\partial x} z - \frac{\partial}{\partial z}(xz + x^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x^3 z + y^3) - \frac{\partial}{\partial y}(xz + x^3)\right]$$

$$= \vec{i}(-x^3) - \vec{j}(-x) + \vec{k}(3x^2 z)$$

$$= \underline{\underline{-x^3\vec{i} + x\vec{j} + 3x^2 z\vec{k}.}}$$

b)





Divergensteoremet :

$$\iint_S \vec{F}\cdot\hat{N}\,dS = \iiint_T \nabla\cdot\vec{F}\,dV$$

$$= \iiint [3(x^2+y^2)+z+1]\,dV \quad \text{(sylinderkoord.)}$$

$$= \int_0^{2\pi}\int_0^{2}\int_{-2}^{2} [3r^2+z+1]\,dz\,r\,dr\,d\theta$$

$$= \iint (3r^2+1)z+\tfrac{1}{2}z^2 \Big|_{-2}^{2} r\,dr\,d\theta$$

$$= \iint [(3r^2+1)(2+2)+\tfrac{1}{2}[2^2-(-2)^2]]\,r\,dr\,d\theta .$$

$$= \iint 4(3r^2+1)\,r\,dr\,d\theta = 4\iint (3r^3+r)\,dr\,d\theta$$

$$= 4\int \tfrac{3}{4}r^4+\tfrac{1}{2}r^2\Big|_0^2 d\theta = 4\int (\tfrac{3}{4}\cdot 2^4+\tfrac{1}{2}\cdot 2^2)\,d\theta =$$

$$= 4\int_0^{2\pi}(3\cdot 4 + 2)\,d\theta = 4\cdot 14\int_0^{2\pi} d\theta = 56\cdot 2\pi$$

$$\underline{\underline{= 112\pi}}$$

c) Beregn $\oint_C \vec{F}\cdot d\vec{r}$.

Skjæringskurven C er sirkelen $x^2 + y^2 = 4$, for $z = 2$.

Bruk Stokes' teorem:

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{N}\,dS.$$

La S være den delen av planet $z=2$ som er avgrenset av sirkelen C.

$\hat{N} = \vec{k}$. $dS = dA =$ flateelementet i sirkelskiva med radius $= 2$.

Bruk polarkoord.

$$\nabla\times\vec{F}\cdot\hat{N} = \nabla\times\vec{F}\cdot\vec{k} = [-x^3\vec{i} + x\vec{j} + 3x^2 z\vec{k}]\cdot\vec{k}$$
$$= 3x^2 z.$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{N}\,dS = \int_S\int 3x^2 z\; r\,dr\,d\theta =$$

$$\underset{\overset{\uparrow}{z=2}}{=} 3 \iint x^2 \cdot 2\, r\,dr\,d\theta = 6 \int_0^{2\pi}\int_0^2 r^2 \cos^2\theta \, r\,dr\,d\theta$$

$$= 6 \int_0^{2\pi}\int_0^2 r^3 \cos^2\theta \,dr\,d\theta = 6 \int \frac{1}{4} r^4 \Big|_0^2 \cos\theta$$

$$= \frac{3}{2} \cdot 2^4 \int \cos^2\theta \,d\theta = 3 \cdot 2^3 \int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$= 3 \cdot 2^2 \int (1+\cos 2\theta)\,d\theta = 12\left[\theta + \frac{1}{2}\underset{\overset{\shortparallel}{0}}{\sin(2\theta)}\right]_0^{2\pi}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1 - \cos^2\theta)$$

$$= 2\cos^2\theta - 1 \Rightarrow$$

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= 12 \cdot 2\pi = \underline{\underline{24\pi}}$$

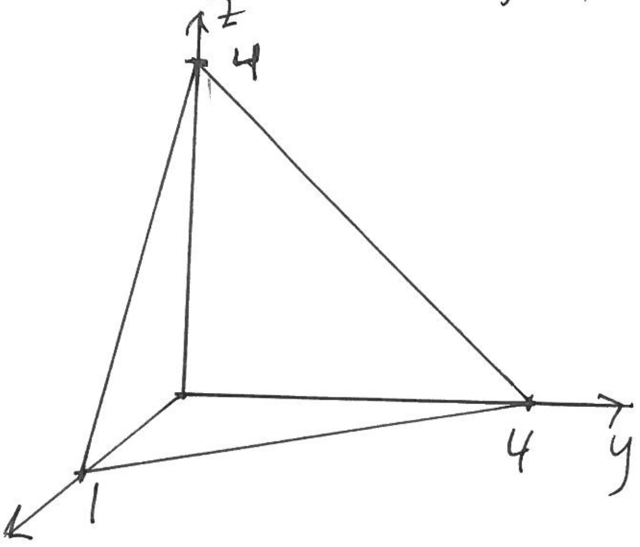
Oppgave 3.

Planet $4x+y+z=4$. Skisser planet (og T):

$x = y = 0 \Rightarrow z = 4$

$x = z = 0 \Rightarrow y = 4$

$y = z = 0 \Rightarrow x = 1$



a) (i). $\vec{n}$ = normalvektor til planet $4x+y+z=4$: $\vec{n}=[4,1,1]$.

Denne vektoren peker oppover, og ut fra T.

$$|\vec{n}|=\sqrt{4^2+1+1}=\sqrt{18}=\sqrt{9\cdot 2}=\underline{3\sqrt{2}}$$

Enhetsnormalvektor:

$$\hat{N}=\frac{\vec{n}}{|\vec{n}|}=\underline{\underline{\frac{[4,1,1]}{3\sqrt{2}}}}$$

(ii) $\text{Fluks}=\iint\limits_S \vec{F}\cdot\hat{N}\,dS=\iint \vec{F}\cdot\hat{N}\,\frac{|\nabla g|}{|\nabla g\cdot\vec{P}|}\,dA$.

$$\vec{F}=yz\vec{i}-3yz\vec{j}+(x-yz)\vec{k}$$

Projeksjonsplan: xy-planet $\Rightarrow \vec{P}=\vec{k}$.

La $\nabla g=\vec{n}$ være normalvektor (i formel).

$\Rightarrow dS=\frac{|\vec{n}|}{|\vec{n}\cdot\vec{k}|}dA=|\vec{n}|dA$, der dA er flateelementet i xy-planet.

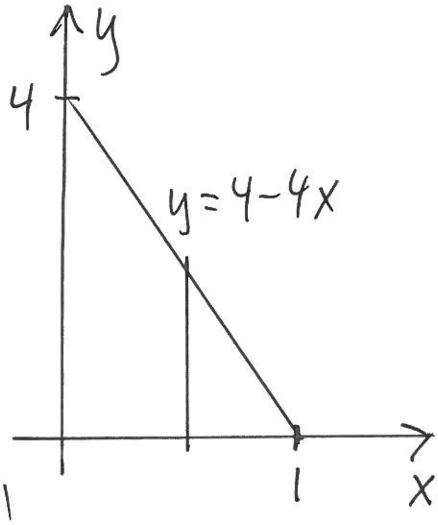
$$\vec{F}\cdot\hat{N}\,dS=\vec{F}\cdot\frac{\vec{n}}{|\vec{n}|}|\vec{n}|dA=\vec{F}\cdot\vec{n}\,dA$$

$$=[yz,-3yz,(x-yz)]\cdot[4,1,1]dA$$

$$=[4yz-3yz+x-yz]dA=x\,dA.$$

Projeksjonen av S: $z = 0 \Rightarrow$

$4x + y = 4 \Rightarrow y = 4 - 4x.$



$$\iint \vec{F}\cdot\hat{N}\,dS$$

$$= \iint x\,dA$$

$$= \int_0^1 \int_0^{4-4x} x\,dy\,dx$$

$$= \int_0^1 xy\Big|_0^{4-4x} dx = \int_0^1 x(4-4x)\,dx$$

$$= \int (4x - 4x^2)\,dx = 2x - \frac{4}{3}x^3\Big|_0^1 = 2 - \frac{4}{3}$$

$$= \frac{6-4}{3} = \underline{\underline{\frac{2}{3}}}$$

b) Beregn $\iiint_T y\,dV$.

Integrasjon i z-retning først.
Deretter over projeksjonen i xy-planet, som i a).

$$\iiint_T y\,dV = \int_0^1 \int_0^{4-4x} \int_0^{4-4x-y} y\,dz\,dy\,dx$$

$$= \iint yz \Big|_0^{4-4x-y} dy\,dx$$

$$= \iint y\,[4-4x-y]\,dy\,dx = \iint [4y-4xy-y^2]\,dy\,dx$$

$$= \int 2y^2 - 2xy^2 - \tfrac{1}{3}y^3 \Big|_0^{4-4x} dx$$

$$= \int y^2 [2-2x-\tfrac{1}{3}y]\,dx$$

$$= \int (4-4x)^2 [2-2x-\tfrac{1}{3}(4-4x)]\,dx$$

$$= \int (4-4x)^2 \left[\tfrac{2}{3}-\tfrac{2}{3}x\right] dx = \tfrac{1}{3}\int (4-4x)^2 (2-2x)\,dx$$

$$= \tfrac{1}{3}\int (4-4x)^2 \cdot \tfrac{1}{2}(4-4x)\,dx = \tfrac{1}{6}\int (4-4x)^3\,dx \,.$$

Subst.:

$u = 4-4x$

$\frac{du}{dx} = -4 \Rightarrow$

$dx = -\frac{du}{4}$

$$= \tfrac{1}{6}\int u^3 \left(-\tfrac{du}{4}\right)$$

$$= -\tfrac{1}{24}\cdot\tfrac{1}{4}u^4 \quad =$$

$$= -\frac{1}{24}\cdot\frac{1}{4}\left(4-4x\right)^4\Big|_0^1$$

$$= -\frac{1}{24}\cdot\frac{1}{4}\left[0-4^4\right] = \frac{4^4}{6\cdot4^2} = \frac{4^2}{6} = \frac{16}{6} = \underline{\underline{\frac{8}{3}}}$$