DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

Universitetet i Stavanger

EKSAMEN I: MAT300 Vektoranalyse

DATO: 13. desember 2014 kl. 0900 - 1300

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII+

OPPGAVESETTET BESTÅR AV 3 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = (t+3)\,\mathbf{i} + 2t\,\mathbf{j} + (2t+1)\,\mathbf{k}; \quad 0 \le t \le 1.$

a) Beregn kurveintegralet

$$\int_C (7xy + y^2 - z)\, ds.$$

b) Vis at vektorfeltet $\mathbf{F}(x,y,z) = z\mathbf{i} + (2yz+1)\,\mathbf{j} + (x+y^2)\,\mathbf{k}$ er konservativt.

c) Finn en potensialfunksjon (skalarfelt) til vektorfeltet $\mathbf{F}$.

d) Beregn kurveintegralet

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 2

Gitt transformasjonen $u = x - y$; $\quad v = 3x + y$, som gir sammenheng mellom (u,v) -koordinatene og (x,y) -koordinatene.
Gitt det endelige området R i xy -planet avgrenset av de rette linjene $y = x$, $y = x+3$, $y = -3x$, og $y = -3x + 6$.

a) Finn Jacobi-determinanten

$$\frac{\partial(x,y)}{\partial(u,v)}$$

b) Skisser området R i xy -planet. Skisser avbildningen av R i uv -planet.

c) Beregn dobbelintegralet

$$\iint_R 4(x-y)(3x+y+1)\,dA$$

der R er området i xy -planet gitt ovenfor, ved å benytte skifte av variable gitt ovenfor.

OPPGAVE 3

Gitt vektorfeltet $\mathbf{F}(x, y, z) = (2y^2 - x^2)\mathbf{i} + 2xy\,\mathbf{j} + z^2\,\mathbf{k}$.
La D være området avgrenset av paraboloiden $z = 6 - x^2 - y^2$ og $xy-$ planet ($z \geq 0$).

a) Finn $div\,\mathbf{F}$ og $curl\,\mathbf{F}$.

b) Bruk divergensteoremet til å beregne flateintegralet

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS,$$

der S er randen (overflaten) til området D, og $\hat{\mathbf{N}}$ er enhetsnormalvektor til S.
$\hat{\mathbf{N}}$ peker utover (fra D).

c) La C være skjæringskurven mellom paraboloiden $z = 6 - x^2 - y^2$ og $xy-$ planet.
Omløpsretning langs C (orientering) er mot urviser, sett ovenfra.
Beregn kurveintegralet

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $\quad a \leq t \leq b$:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\frac{d\mathbf{r}}{dt}|\,dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}}\,ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})\,dt.$$

Flateintegral av en funksjon f over en flate $S: g(x,y,z) = K$ (K er en konstant):

$$\iint_S f\,dS = \iint_R f\,\frac{|\nabla g|}{|\nabla g \cdot \mathbf{P}|}\,dA.$$

Stokes' teorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\,dS = \iint_R (\nabla \times \mathbf{F}) \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{P}|}\,dA.$$

Divergensteoremet:

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\,dS.$$

$$div\,\mathbf{F} = \nabla \cdot \mathbf{F}, \quad curl\,\mathbf{F} = \nabla \times \mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.
Kulekoordinater: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

MAT 300 vektoranalyse
Eksamen 13. desember 2014
Løsning

Oppgave 1.

$$C: \vec{r}(t) = (t+3)\vec{i} + 2t\vec{j} + (2t+1)\vec{k}$$
$$0 \le t \le 1.$$

a) Beregn kurveintegralet

$$\int_C (7xy + y^2 - z)\,ds.$$

C: $x = t+3$, $y = 2t$, $z = 2t+1$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 2.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{1 + 2^2 + 2^2} = \sqrt{9} = 3.$$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt = 3\,dt.$$

$$\int_C (7xy + y^2 - z)\,ds = \int_0^1 \left[7(t+3)\cdot 2t + 4t^2 - (2t+1)\right] 3\,dt$$

$$= 3\int \left(14t^2 + 42t + 4t^2 - 2t - 1\right) dt =$$

$$3\int_0^1 (18t^2 + 40t - 1)\,dt = 3\left[\frac{18}{3}t^3 + \frac{40}{2}t^2 - t\right]_0^1$$

$$= 3\left[6t^3 + 20t^2 - t\right]_0^1 = 3\left[6 + 20 - 1\right] = 3 \cdot 25$$

$$\underline{\underline{= 75}}$$

b) $\vec{F}(x, y, z) = z\vec{i} + (2yz + 1)\vec{j} + (x + y^2)\vec{k}$

Vis at $\vec{F}$ er konservativ:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 2yz+1 & x+y^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x + y^2) - \frac{\partial}{\partial z}(2yz+1)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial z} z\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(2yz+1) - \frac{\partial}{\partial y} z\right]$$

$$= \vec{i}\,(2y - 2y) - \vec{j}\,(1 - 1) + \vec{k} \cdot 0 = \vec{0}.$$

$\nabla \times \vec{F} = \vec{0} \Rightarrow$ $\underline{\underline{\vec{F} \text{ er konservativ}}}$.

c) Finn en potensialfunksjon φ:

$$(\vec{F} = \nabla\varphi)$$

$$\frac{\partial\varphi}{\partial x} = z \qquad \text{I}$$

$$\frac{\partial\varphi}{\partial y} = 2yz+1 \qquad \text{II}$$

$$\frac{\partial\varphi}{\partial z} = x+y^2 \qquad \text{III}$$

$$\text{I} \quad \frac{\partial\varphi}{\partial x} = z \Rightarrow \varphi = \int z\,dx = \underline{xz + C_1(y,z)}$$

Deriver mhp. y:

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\left(xz + C_1(y,z)\right) = \frac{\partial C_1}{\partial y} \underset{\text{II}}{=} 2yz+1$$

$$\Rightarrow \underline{C_1(y,z)} = \int(2yz+1)\,dy = \underline{y^2z + y + C_2(z)}$$

$$\Rightarrow \underline{\varphi = xz + y^2z + y + C_2(z)}.$$

Deriver mhp. z:

$$\frac{\partial\varphi}{\partial z} = x + y^2 + C_2' \underset{\text{III}}{=} x+y^2 \Rightarrow C_2' = 0$$

$$\Rightarrow C_2 = \text{konst.}$$

$$\Rightarrow \underline{\underline{\varphi = xz + y + y^2z}}$$

d) Beregn $\int_C \vec{F} \cdot d\vec{r}$.

Benytt: $\vec{F}$ er konservativ.

$C: x = t+3, \quad y = 2t, \quad z = 2t+1$

$A: t=0: x = 3, \quad y = 0, \quad z = 1.$

$\vec{r}(0) = [3, 0, 1]$

$B: t=1: x = 4, \quad y = 2, \quad z = 3.$

$\vec{r}(1) = [4, 2, 3].$

$$\int_C \vec{F} \cdot d\vec{r} = \varphi \Big|_{\vec{r}(0)}^{\vec{r}(1)} = xz + y + y^2 z \Big|_{(3,0,1)}^{(4,2,3)}$$

$$= 4 \cdot 3 + 2 + 2^2 \cdot 3 - (3 \cdot 1 - 0)$$

$$= 12 + 2 + 12 - 3 = \underline{\underline{23}}$$

Oppgave 2

Transformasjon: $u = x - y, \quad v = 3x + y.$

a) Finn $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 1 + 3 = 4.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \underline{\underline{\frac{1}{4}}}$$

Alternativ: Løs mhp. x og y:

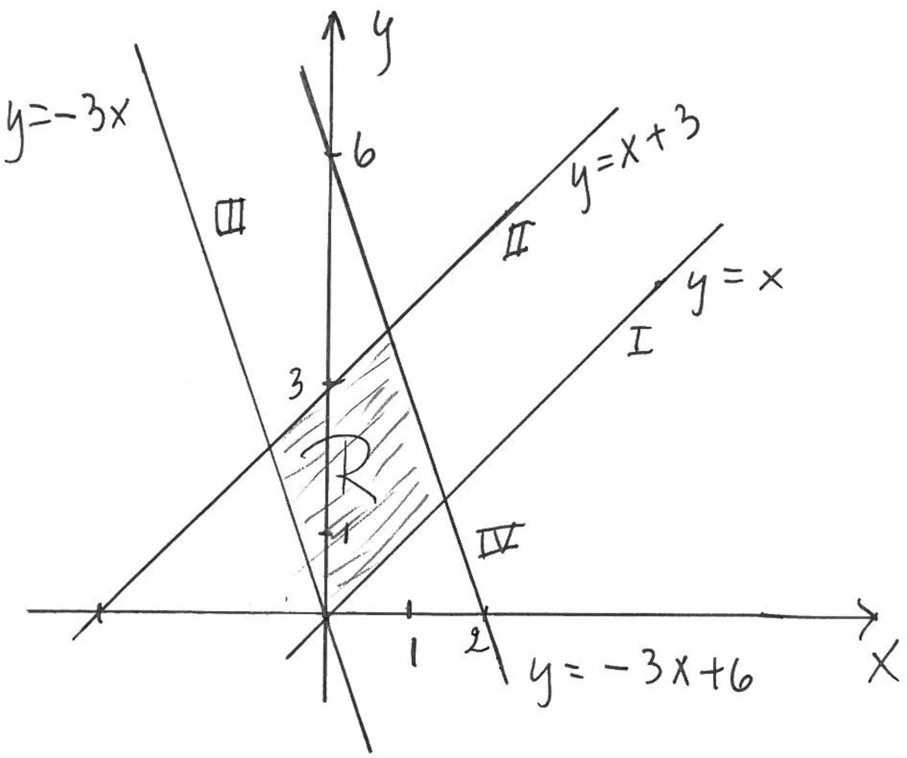
$$\begin{aligned} x - y &= u \qquad \mathrm{I} \\ 3x + y &= v \qquad \mathrm{II} \end{aligned}$$

$\mathrm{I}+\mathrm{II}: 4x = u+v \Rightarrow x = \frac{1}{4}(u+v)$

$-3\mathrm{I}+\mathrm{II}: 4y = -3u+v \Rightarrow y = \frac{1}{4}(-3u+v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{16} + \frac{3}{16} = \underline{\frac{1}{4}}$$

b) R: $y = x$, $y = x+3$, $y = -3x$, $y = -3x+6$.



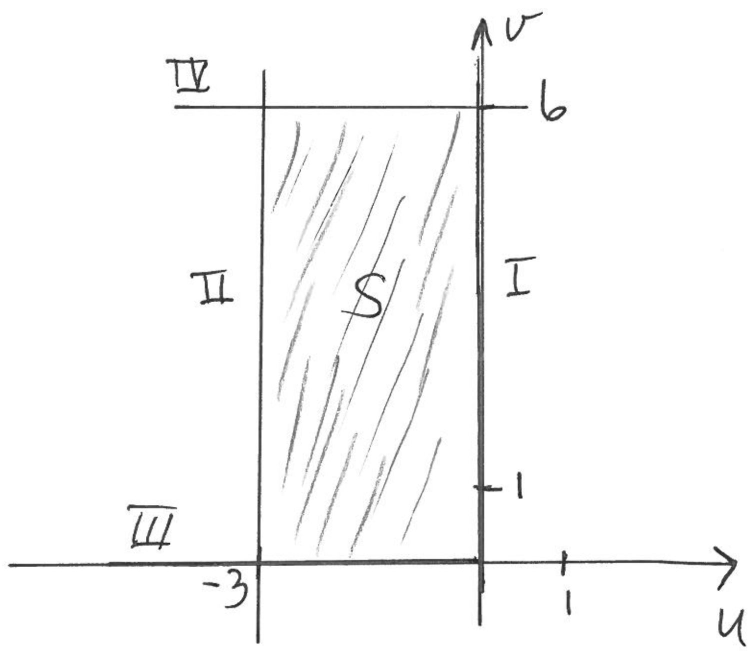
Avbildning i uv-planet:

I : $y = x \Rightarrow x - y = 0 \Rightarrow \underline{u = 0}$

II : $y = x + 3 \Rightarrow x - y = -3 \Rightarrow \underline{u = -3}$

III : $y = -3x \Rightarrow 3x + y = 0 \Rightarrow \underline{v = 0}$

IV : $y = -3x + 6 \Rightarrow 3x + y = 6 \Rightarrow \underline{v = 6}$.



Alternativ:

I : $y = x \Rightarrow \frac{1}{4}(-3u + v) = \frac{1}{4}(u + v) \Rightarrow -4u = 0$

$\Rightarrow \underline{u = 0}$.

II : $y = x + 3 \Rightarrow \frac{1}{4}(-3u + v) = \frac{1}{4}(u + v) + 3 \Rightarrow$

$-3u + v - u - v = 12 \Rightarrow -4u = 12 \Rightarrow \underline{u = -3}$

III : $y = -3x \Rightarrow \frac{1}{4}(-3u + v) = -3 \cdot \frac{1}{4}(u + v) \Rightarrow$

$-3u + v = -3u - 3v \Rightarrow 4v = 0 \Rightarrow \underline{v = 0}$

IV : $y = -3x + 6 \Rightarrow \frac{1}{4}(-3u + v) = -3 \cdot \frac{1}{4}(u + v) + 6 \Rightarrow$

$-3u + v = -3u - 3v + 24 \Rightarrow 4v = 24 \Rightarrow \underline{v = 6}$

c)

$$\iint_R 4(x-y)(3x+y+1)\,dA$$

$$= 4\iint_S u(v+1)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= 4\int_0^6\int_{-3}^0 u(v+1)\cdot\frac{1}{4}\,du\,dv$$

$$= \int_0^6 \frac{1}{2}u^2\Big|_{-3}^0 (v+1)\,dv = \frac{1}{2}\int(0-(-3)^2)(v+1)\,dv$$

$$= -\frac{9}{2}\int(v+1)\,dv = -\frac{9}{2}\left[\frac{1}{2}v^2+v\right]_0^6$$

$$= -\frac{9}{2}\left[\frac{1}{2}6^2+6-0\right] = -\frac{9}{2}(18+6) = -\frac{9}{2}\cdot 24$$

$$= -9\cdot 12 = \underline{\underline{-108}}$$

Oppgave 3

$$\vec{F}(x,y,z) = (2y^2 - x^2)\vec{i} + 2xy\vec{j} + z^2\vec{k}$$

a) $\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2y^2 - x^2) + \frac{\partial}{\partial y}2xy + \frac{\partial}{\partial z}z^2$

$$= -2x + 2x + 2z = \underline{\underline{2z}}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 - x^2 & 2xy & z^2 \end{vmatrix}$$

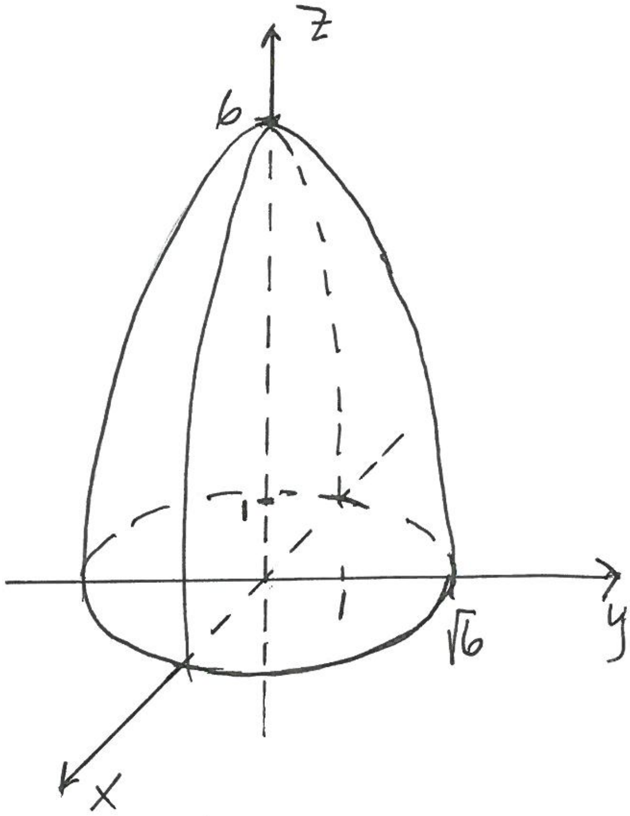
$$= \vec{i}\left[\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}2xy\right] - \vec{j}\left[\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial z}(2y^2 - x^2)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}2xy - \frac{\partial}{\partial y}(2y^2 - x^2)\right]$$
$$= \vec{i}\cdot 0 - \vec{j}\cdot 0 + \vec{k}[2y - 4y] = \underline{\underline{-2y\vec{k}}}$$

b) Paraboloiden $z = 6 - x^2 - y^2 = 6 - r^2$
i sylinderkoord.

Skisser: $z=0$: $x^2 + y^2 = 6$, sirkel med radius $= \sqrt{6}$.

$x = 0$: $z = 6 - y^2$, parabel

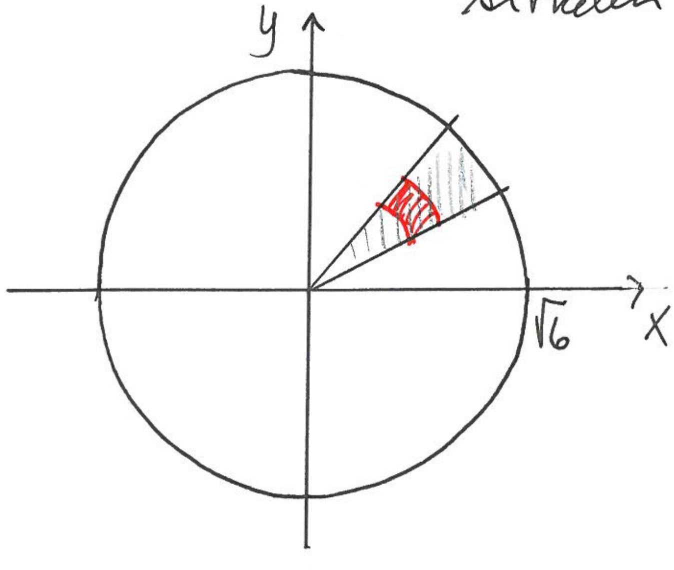
$y = 0$: $z = 6 - x^2$ — " —



Divergensteoremet : $\iint_S \vec{F} \cdot \hat{N} \, dS = \iiint_D \nabla \cdot \vec{F} \, dV$

Beregn $\iiint_D \nabla \cdot \vec{F} \, dV$.

Skjøring mellom paraboloiden og xy-planet : $z = 0 \Rightarrow x^2 + y^2 = 6$, sirkelen ovenfor.



$$\iiint_D \nabla\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\sqrt{6}}\int_0^{6-r^2} 2z\;dz\,r\,dr\,d\theta$$

(cylinderkoord.)

$$= \iint z^2\Big|_0^{6-r^2} r\,dr\,d\theta \;= \iint (6-r^2)^2\, r\,dr\,d\theta .$$

Subst.:

$u = 6-r^2$

$\frac{du}{dr} = -2r$

$dr = -\frac{du}{2r}$

$$= \iint u^2\, r\left(-\frac{du}{2r}\right) d\theta$$

$$= -\frac{1}{2}\iint u^2\,du\,d\theta$$

$$= -\frac{1}{2}\int \frac{1}{3}u^3\Big|\,d\theta$$

$$= -\frac{1}{6}(6-r^2)^3\Big|_0^{\sqrt{6}}\int d\theta$$

$$= -\frac{1}{6}\left[0-6^3\right]\cdot 2\pi \;= \frac{1}{6}\cdot 6^3\cdot 2\pi = 6^2\cdot 2\pi$$

$$\underline{\underline{= 72\pi}}$$

Altså $\iint_S \vec{F}\cdot\hat{N}\,dS = \underline{\underline{72\pi}}$

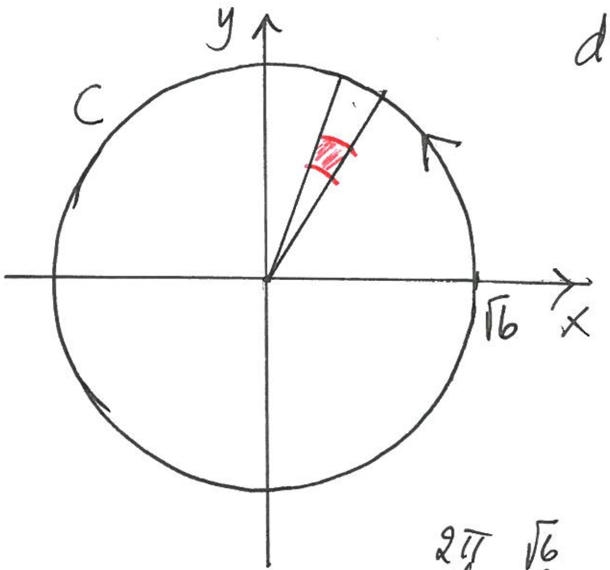
c) Beregn $\oint_C \vec{F} \cdot d\vec{r}$.

Bruk Stokes teorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{N}\, dS$$

der S er en flate som har C til randkurve.

La S være sirkelskiva $S_1: x^2+y^2 \le 6$ i xy-planet. $\hat{N} = \vec{k}$.



$dS = dA =$ flateelement i xy-planet.

Bruk polarkoord.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S_1} \nabla \times \vec{F} \cdot \vec{k}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{6}} -2y\vec{k} \cdot \vec{k}\, dA = -2\int_0^{2\pi}\int_0^{\sqrt{6}} y\, dA$$

$$= -2\iint r \sin\theta \; r\,dr\,d\theta$$

$$= -2 \iint r^2 \sin\theta \,dr\,d\theta = -2\int \frac{1}{3} r^3 \Big|_0^{\sqrt{6}} \sin\theta\, d\theta$$

$$= -\frac{2}{3}(\sqrt{6})^3 \left(-\cos\theta\right)\Big|_0^{2\pi}$$

$$= \frac{2}{3}\, 6\sqrt{6}\,[\cos 2\pi - \cos 0] = 4\sqrt{6}\,[1-1] = \underline{\underline{0.}}$$

<u>Alternativ</u>: Symmetribetrakning:

$\iint y\,dA = 0$, fordi integrasjon av oddefunksjonen y over det symmetriske området i y-retning

$\Rightarrow \iint y\,dA = 0.$