DET TEKNISK-NATURVITENSKAPELIGE FAKULTET

Universitetet i Stavanger

EKSAMEN I: MAT300 Vektoranalyse

DATO: 3. mars 2015 kl. 0900 - 1300

TILLATTE HJELPEMIDLER:
Rottmann: Matematisk formelsamling
Kalkulatorer: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII+

OPPGAVESETTET BESTÅR AV 4 OPPGAVER PÅ 2 SIDER + 1 SIDE MED FORMLER

OPPGAVE 1

Gitt kurven C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 4t\,\mathbf{k}$; $0 \leq t \leq \pi$.

a) Finn enhetstangentvektor til C i punktet svarende til $t = \pi/2$.

b) Beregn kurveintegralet

$$\int_C (x + \sqrt{5}\,z^3)\,ds.$$

Gitt vektorfeltet $\mathbf{F}(x,y,z) = x\mathbf{i} + y\,\mathbf{j} + y\,\mathbf{k}$.

c) Beregn kurveintegralet

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

OPPGAVE 2

Gitt følgende randverdiproblem:

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \quad \text{der} \quad u = u(x,t), \quad 0 < x < \pi, \quad t > 0.$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0.$$

$$u(x,0) = 3\sin(2x) - 2\sin(5x), \quad 0 < x < \pi.$$

Vis at funksjonen $z(x,t)$ er en funksjon som tilfredsstiller randverdiproblemet ovenfor:

$$z(x,t) = 3e^{-20t}\sin(2x) - 2e^{-125t}\sin(5x).$$

OPPGAVE 3

Gitt vektorfeltet $\mathbf{F}(x, y, z) = xy^2z\mathbf{i} + x^2yz\,\mathbf{j} + z^2\,\mathbf{k}$.
La T være området avgrenset av sylinderen $x^2 + y^2 = 9$, xy -planet og planet $z = 2$.

a) Finn $div\,\mathbf{F}$ og $curl\,\mathbf{F}$.

b) Bruk divergensteoremet til å beregne flateintegralet

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

der S er randen (overflaten) til området T. $\hat{\mathbf{N}}$ er enhetsnormalvektor til S, og peker utover (fra T).

OPPGAVE 4

La S være den delen av kuleflaten $x^2 + y^2 + z^2 = 25$ som ligger over planet $z = 3$, og la T være området avgrenset av flaten S og planet $z = 3$.

a) Beregn

$$\iiint_T z\, dV.$$

b) Beregn fluks gjennom S

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

av vektorfeltet $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\,\mathbf{j} - 2xy\,\mathbf{k}$.
$\hat{\mathbf{N}}$ er enhetsnormalvektor til S, og peker utover (fra T).

c) Beregn flateintegralet

$$\iint_S z^2\, dS$$

Lykke til!

Formler:

Kurveintegral av en funksjon f langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $\quad a \le t \le b$:

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\left|\frac{d\mathbf{r}}{dt}\right| dt.$$

Kurveintegral av et vektorfelt $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, langs en kurve C: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_C \mathbf{F}\cdot\hat{\mathbf{T}}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b (F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt})\,dt.$$

Flateintegral av en funksjon f over en flate $S : g(x,y,z) = K$ (K er en konstant):

$$\iint_S f\,dS = \iint_R f\,\frac{|\nabla g|}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Stokes' teorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{N}}\,dS = \iint_R (\nabla\times\mathbf{F})\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{P}|}\,dA.$$

Divergensteoremet:

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS.$$

$$div\,\mathbf{F} = \nabla\cdot\mathbf{F}, \quad curl\,\mathbf{F} = \nabla\times\mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Sylinderkoordinater: $(r\cos\theta, r\sin\theta, z) = (x,y,z)$.
Kulekoordinater: $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) = (x,y,z)$.

MAT 300 Vektoranalyse

Eksamen 3. mars 2015

Lösning

Oppgave 1.

Kurven C: $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 4t\vec{k}$

$0 \le t \le \pi$.

Dvs.: $x = 2\cos t$, $y = 2\sin t$, $z = 4t$

$\Rightarrow \frac{dx}{dt} = -2\sin t$, $\frac{dy}{dt} = 2\cos t$, $\frac{dz}{dt} = 4$.

a) Enhetstangentvektor $\hat{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$= -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 4\vec{k}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 4^2}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t) + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{2}} = -2\sin\frac{\pi}{2}\,\vec{i} + 2\cos\frac{\pi}{2}\,\vec{j} + 4\vec{k}$$

$$= -2\vec{i} + 4\vec{k}$$

$$\hat{T} = \frac{-2\vec{i} + 4\vec{k}}{2\sqrt{5}} = \underline{\underline{\frac{-\vec{i} + 2\vec{k}}{\sqrt{5}}}}$$

b) $ds = \left|\frac{d\vec{r}}{dt}\right| dt = 2\sqrt{5}\, dt.$

$$\int_C (x + \sqrt{5}\ z^3)\, ds = 2\sqrt{5} \int (x + \sqrt{5}\, z^3)\, dt$$

$$= 2\sqrt{5} \int (2\cos t + \sqrt{5} \cdot 4^3 t^3)\, dt$$

$$= 2\sqrt{5} \left[2\sin t + \sqrt{5} \cdot 4^2 t^4\right]_0^{\pi}$$

$$= 2\sqrt{5} \left[2 \underset{0}{\underset{\shortparallel}{\sin\pi}} + \sqrt{5} \cdot 16 \cdot \pi^4 - 0\right]$$

$$= 2\sqrt{5} \cdot \sqrt{5} \cdot 16\pi^4 = 32 \cdot 5 \cdot \pi^4 = \underline{\underline{160\pi^4}}$$

c) $\vec{F} = x\vec{i} + y\vec{j} + y\vec{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt$$

$$= \int_0^{\pi} \left(x \frac{dx}{dt} + y \frac{dy}{dt} + y \frac{dz}{dt}\right) dt$$

$$= \int \left[2\cos t(-2\sin t) + 2\sin t \cdot 2\cos t + 2\sin t \cdot 4\right] dt$$

$$= \int \left[-4\cos t \sin t + 4\cos t \sin t + 8\sin t\right] dt$$

$$= 8 \int_0^{\pi} \sin t\, dt = 8(-\cos t)_0^{\pi} = -8[\cos\pi - \cos 0]$$

$$= -8[-1-1] = \underline{\underline{16}}$$

Oppgave 2.

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = 3\sin(2x) - 2\sin(5x).$$

Vis at $z(x,t) = 3e^{-20t}\sin(2x) - 2e^{-125t}\sin(5x)$ tilfredsstiller randverdiproblemet.

Vis at $z(x,t)$ er løsning av den partielle diff. ligningen, ved innsetting:

Venstre side:

$$\frac{\partial z}{\partial t} = 3\cdot(-20)e^{-20t}\sin(2x) - 2(-125)e^{-125t}\sin(5x)$$

$$= \underline{-60e^{-20t}\sin(2x) + 250e^{-125t}\sin(5x)}$$

Høyre side:

$$\frac{\partial z}{\partial x} = 3\cdot 2e^{-20t}\cos(2x) - 2\cdot 5e^{-125t}\cos(5x)$$

$$\frac{\partial^2 z}{\partial x^2} = -6\cdot 2e^{-20t}\sin(2x) + 10\cdot 5e^{-125t}\sin(5x)$$

$$5\frac{\partial^2 z}{\partial x^2} = -12\cdot 5e^{-20t}\sin(2x) + 50\cdot 5e^{-125t}\sin(5x)$$

$$= \underline{-60e^{-20t}\sin(2x) + 250e^{-125t}\sin(5x)}$$

OK.

Sett $x=0$ i z:

$$z(0,t) = 3e^{-20t}\sin 0 - 2e^{-125t}\sin 0 = 0.$$

OK

Sett $x=\pi$ i z:

$$z(\pi,t) = 3e^{-20t}\sin(2\pi) - 2e^{-125t}\sin(5\pi) = 0.$$

OK.

Sett $t=0$ i z:

$$z(x,0) = 3e^{0}\sin(2x) - 2e^{0}\sin(5x)$$

$$= \underline{3\sin(2x) - 2\sin(5x)} = \underline{u(x,0)}$$

OK.

Oppgave 3

$$F(x,y,z) = xy^2z\,\vec{i} + x^2yz\,\vec{j} + z^2\vec{k}$$

a) $$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xy^2z) + \frac{\partial}{\partial y}(x^2yz) + \frac{\partial}{\partial z}z^2$$

$$= y^2z + x^2z + 2z = \underline{\underline{z(x^2+y^2)+2z}}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z & x^2yz & z^2 \end{vmatrix}$$

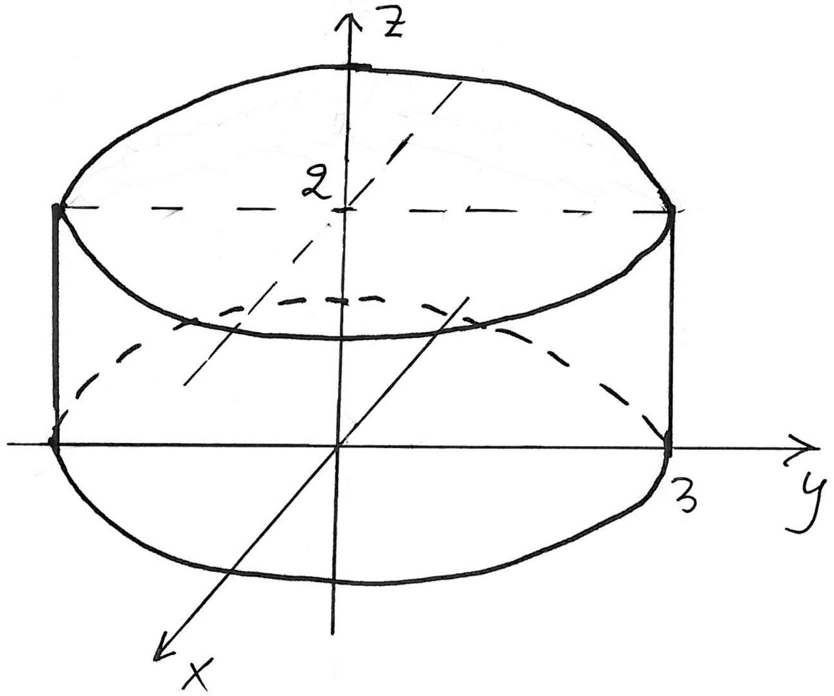
$$= \vec{i}\left(\frac{\partial}{\partial y}z^2 - \frac{\partial}{\partial z}x^2yz\right) - \vec{j}\left[\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial z}xy^2z\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}x^2yz - \frac{\partial}{\partial y}xy^2z\right] =$$

$$= -x^2y\vec{i} + xy^2\vec{j} + (2xyz - 2xyz)\vec{k}$$

$$= \underline{\underline{-x^2y\vec{i} + xy^2\vec{j}}}$$

b)



Divergensteoremet :

$$\iint_S \vec{F}\cdot\hat{N}\,dS = \iiint_T \nabla\cdot\vec{F}\,dV$$

$$= \iiint_T \left((x^2+y^2)z + 2z\right)dV$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (r^2z + 2z)\,dz\,r\,dr\,d\theta \quad =$$

sylinder-koord.

$$\iint r^2 \tfrac{1}{2} z^2 + z^2 \Big|_0^2 \, r\,dr\,d\theta$$

$$= \iint \left(r^2 \cdot \tfrac{1}{2} \cdot 2^2 + 2^2 - 0\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} \left(2r^2 + 4\right) r\,dr\,d\theta = \iint \left(2r^3 + 4r\right) dr\,d\theta$$

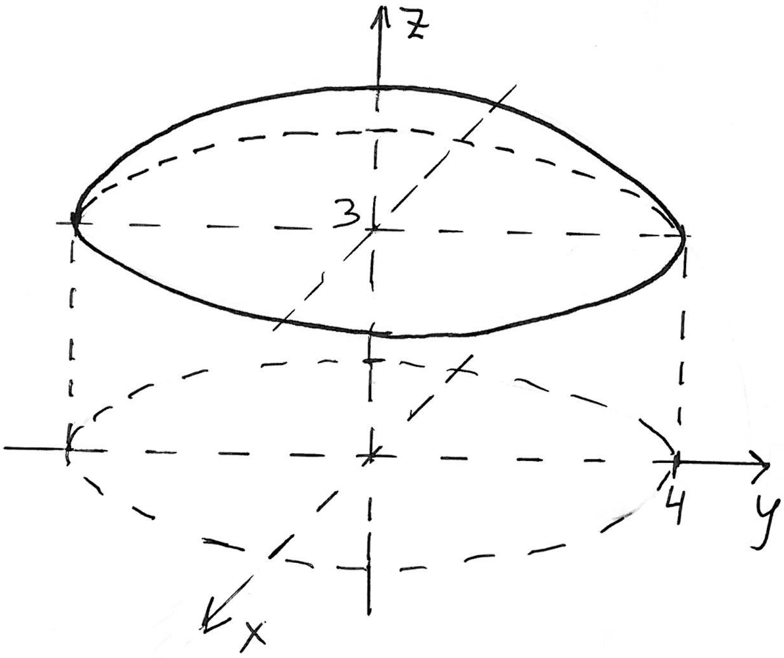
$$= \int_0^{2\pi} \tfrac{2}{4} r^4 + \tfrac{4}{2} r^2 \Big|_0^3 \, d\theta = \int \left(\tfrac{1}{2} \cdot 3^4 + 2 \cdot 3^2 - 0\right) d\theta$$

$$= 2\pi \cdot 3^2 \left[\tfrac{1}{2} \cdot 3^2 + 2\right] = 18\pi \left[\tfrac{9}{2} + 2\right]$$

$$= 18\pi \cdot \tfrac{13}{2} = 9\pi \cdot 13 = \underline{\underline{117\pi}}$$

<u>Oppgave 4.</u>

$S: x^2 + y^2 + z^2 = 25, \quad z \geq 3.$



Skjæring kuleflate og $z=3$:

$$x^2+y^2+3^2=25 \Rightarrow \underline{x^2+y^2} = 25-9 = \underline{16}.$$

Sirkel med radius $= 4$.

a) Beregn $\iiint_T z\,dV$.

$$x^2+y^2+z^2=25 \Rightarrow r^2+z^2=25, \text{ sylinderkoord.}$$

$$\Rightarrow z=\sqrt{25-r^2}, \text{ på kuleflata.} \quad (z \geq 0).$$

$$\iiint_T z\,dV = \int_0^{2\pi}\int_0^4\int_3^{\sqrt{25-r^2}} z\,dz\,r\,dr\,d\theta$$

$$= \iint \frac{1}{2}z^2\Big|_3^{\sqrt{25-r^2}} r\,dr\,d\theta$$

$$= \frac{1}{2}\iint\left(25-r^2-3^2\right) r\,dr\,d\theta = \frac{1}{2}\iint(16-r^2)\,r\,dr\,d\theta$$

$$= \frac{1}{2}\iint(16r-r^3)\,dr\,d\theta = \frac{1}{2}\int \frac{16}{2}r^2-\frac{1}{4}r^4\Big|_0^4\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(8\cdot 4^2-\frac{1}{4}\cdot 4^4\right)d\theta = 2\pi\cdot\frac{1}{2}\cdot 4^2\,(8-4)$$

$$= 16\pi\cdot 4 = \underline{\underline{64\pi}}$$

b)

$$\iint_S \vec{F}\cdot\hat{N}\,dS$$

$$\vec{F} = yz\vec{i} + xz\vec{j} - 2xy\vec{k}$$

Normalvektor til S:

Definer $g(x,y,z) = x^2+y^2+z^2$.

Kuleflata er nivåflata $g = 25$.

$\nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$. (peker oppover)

$$\hat{N} = \frac{\nabla g}{|\nabla g|}$$

$$\vec{F}\cdot\nabla g = [yz\vec{i} + xz\vec{j} - 2xy\vec{k}]\cdot[2x\vec{i} + 2y\vec{j} + 2z\vec{k}]$$

$$= 2xyz + 2xyz - 4xyz = 0$$

$\vec{F}\cdot\nabla g = 0 \Rightarrow \vec{F}\cdot\hat{N} = 0 \Rightarrow$

fluks, $\iint_S \vec{F}\cdot\hat{N}\,dS = 0$

c) Beregn $\iint_S z^2\,dS$.

$$dS = \frac{|\nabla g|}{|\nabla g\cdot\vec{P}|}\,dA.$$

Projeksjon i xy-planet: $\vec{P} = \vec{k} \Rightarrow$

$$dS = \frac{|\nabla g|}{|\nabla g\cdot\vec{k}|}\,dA$$

$$|\nabla g| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{25} = 2\cdot 5 = 10$$

på kuleflata.

$$\nabla g \cdot \vec{k} = 2z \implies$$

$$dS = \frac{10}{2z}\,dA = \frac{5}{z}\,dA \qquad (z \geq 0)$$

$$\iint_S z^2\,dS = \iint z^2 \frac{5}{z}\,dA = 5\iint z\,dA$$

$$= 5\int_0^{2\pi}\int_0^4 z\;r\,dr\,d\theta = 5\int_0^{2\pi}\int_0^4 \sqrt{25-r^2}\;r\,dr\,d\theta$$

Subst.:

$u = 25 - r^2$

$\frac{du}{dr} = -2r$

$dr = -\frac{du}{2r}$

$$= 5\iint u^{1/2}\,r\left(-\frac{du}{2r}\right)d\theta$$

$$= -\frac{5}{2}\iint u^{1/2}\,du\,d\theta$$

$$= -\frac{5}{2}\cdot\frac{2}{3}\int_0^{2\pi} u^{3/2}\,d\theta$$

$$= -\frac{5}{3}\int_0^{2\pi}\left(25-r^2\right)^{3/2}\Big|_0^4\,d\theta$$

$$= -\frac{5}{3}\cdot 2\pi\left[(25-4^2)^{3/2} - 25^{3/2}\right]$$

$$= -\frac{10}{3}\pi\left[9^{3/2} - 5^3\right] = -\frac{10}{3}\left[3^3 - 5^3\right] =$$

$$= -\frac{10\pi}{3}\left[27-125\right] = \frac{10\pi}{3}\cdot 98$$

$$= \frac{980}{3}\pi$$