THE FACULTY OF SCIENCE AND TECHNOLOGY

EXAM I: MAT300 Vectoranalysis

DATE: 14. december 2014, 09:00 - 13:00

PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII

THE EXERCISE SHEET CONTAINS 4 EXERCISES ON 2 PAGES + 1 PAGE WITH FORMULAS

EXERCISE 1

Let C be the curve: $\mathbf{r}(t) = 3\sin t\ \mathbf{i} - 4\sin t\ \mathbf{j} + 5\cos t\ \mathbf{k}; \quad 0 \leq t \leq \pi$.

a) Compute the line integral
$$\int_C (x + yz)ds.$$

b) Prove that the vector field $\mathbf{F}(x, y, z) = (2x + y)\mathbf{i} + (e^z + x)\mathbf{j} + ye^z\mathbf{k}$ is conservative.

c) Compute the line integral
$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

EXERCISE 2

Consider the transformation $x = u - v; \quad y = -2u + v$, between (u, v) -coordinates and (x, y) -coordinates.
Let R be the bounded domain on the plane xy bounded by the lines:
$y = -x,\ y = -x - 1,\ y = -2x,$ og $y = -2x - 1$.

a) Draw R in the plane xy and the image of R in the plane uv.

b) Compute the double integral
$$\iint_R (y + 2x)(x + y + 1)dA,$$
using the above change of coordinates.

EXERCISE 3

Let F be the vector field $\mathbf{F}(x,y,z) = (y+z)\mathbf{i} - (z+6y^2)\mathbf{j} + (z\cos y + y\sin x)\mathbf{k}$.

a) Compute *div* $\mathbf{F}$ and *curl* $\mathbf{F}$.

b) Let C be the curve of intersection of the sphere $x^2+y^2+z^2=5$ with the plane $z=1$. Let C be oriented counterclockwise looking from above. Compute the integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

EXERCISE 4

Let S be the part of the paraboloid $z = 9 - x^2 - y^2$, that lies above the plane xy.
Let T be the domain bounded by the surface S and the paraboloid $z = x^2+y^2-9$.

a) Compute

$$\iiint_T (z+1)dV.$$

b) Consider the vector field $\mathbf{F}(x,y,z) = 2yz^2\mathbf{i} - 2xz^2\mathbf{j} + z\mathbf{k}$.
Compute the flux

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{N}}\,dS$$

of $\mathbf{F}$ out of T, where $\hat{\mathbf{N}}$ is the unit normal vector to S.

Good luck!

Formulas:

Line integral of a function f along a curve C: $\mathbf{r} = \mathbf{r}(t), \quad a \leq t \leq b$:

$$\int_C f ds = \int_a^b f(r(t))|\frac{d\mathbf{r}}{dt}|\ dt.$$

Line integral of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, along a curve C: $\mathbf{r} = \mathbf{r}(t)$, $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 d_y + F_3 dz = \int_a^b (F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}) dt.$$

Surface integral of a function f over a surface $S : g(x, y, z) = K$ (K is a constant):

$$\iint_S f\ dS = \iint_R f \frac{|\nabla g|}{|\nabla g \cdot \mathbf{P}|}\ dA.$$

Stokes' theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\ dS = \iint_S (\nabla \times \mathbf{F}) \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{P}|}\ dA.$$

Divergence theorem:

$$\iiint_D \nabla \cdot \mathbf{F}\ dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\ dS.$$

$div\ \mathbf{F} = \nabla \cdot \mathbf{F}, \quad curl\ \mathbf{F} = \nabla \times \mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x, y, z)$.
Spherical coordinates $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, Rcos\phi) = (x, y, z)$

EXAM

EX. 1.

$$C: \; r(t) = 3\sin t \,\vec{i} - 4\sin t \,\vec{j} + 5\cos t \,\vec{k} \qquad 0 \le t \le \pi$$

a)

$$\int_C (x+yz)\, ds = \int_0^{\pi} \left(3\sin t - (4\sin t)(5\cos t)\right) \cdot \left|\frac{d\vec{r}}{dt}\right| dt =$$

$$= \int_0^{\pi} \left(3\sin t - 20\sin t\cos t\right) \cdot \sqrt{(3\cos t)^2 + (4\cos t)^2 + (5\sin t)^2}\; dt =$$

$$= 5\left(\int_0^{\pi} 3\sin t\, dt - \int_0^{\pi} 20\sin t\cos t\, dt\right) = -15\cos t\Big|_0^{\pi} - 10\sin^2 t\Big|_0^{\pi} =$$

$$-15(-1-1) - 10(0-0) = 30$$

b) $\vec{F}(x,y,z) = (2x+y)\vec{i} + (e^z + x)\vec{j} + ye^z\,\vec{k}$

LOOK FOR POTENTIAL φ SUCH THAT $\nabla\varphi = \vec{F}$

$$\frac{\partial\varphi}{\partial x} = 2x+y \Rightarrow \varphi = \int 2x+y\, dx = x^2 + xy + C_1(y,z) \Rightarrow \frac{\partial\varphi}{\partial y} = \cancel{x} + \frac{\partial C_1}{\partial y} = e^z + x$$

$$\Rightarrow C_1 = \int e^z\, dy \Rightarrow C_1(y,z) = ye^z + C_2(z) \Rightarrow \varphi = x^2 + xy + ye^z + C_2(z)$$

$$\Rightarrow \frac{\partial\varphi}{\partial z} = ye^z + C_2'(z) \Rightarrow C_2 = \int 0\, dz \Rightarrow C_2(z) = C_3 \text{ - CONSTANT} \Rightarrow$$

$$\varphi = x^2 + xy + ye^z + C_3$$

CANDIDATE FOR POTENTIAL: $\varphi = x^2 + xy + ye^z$

CHECK THAT IT IS A POTENTIAL:

$$\frac{\partial\varphi}{\partial x} = 2x+y = F_1, \quad \frac{\partial\varphi}{\partial y} = x + e^z = F_2, \quad \frac{\partial\varphi}{\partial z} = ye^z = F_3$$

I.E. $\vec{F} = \nabla\varphi$, IN PARTICULAR $\vec{F}$ IS CONSERVATIVE.

c)

$$\int_C F\cdot d\vec{r} \underset{\uparrow}{=} \varphi(r(\pi)) - \varphi(r(0)) = \varphi((0,0,-5)) - \varphi((0,0,5)) =$$

F - CONSERVATIVE WITH φ - POTENTIAL

$$0 - 0 = 0$$

COMMENTS CONCERNING PROBLEMS THAT I FOUND IN YOUR SOLUTIONS:

a) WRONG COMPUTATION OF INTEGRAL $\int \sin t \cos t \, dt$, WRONG ASSIGNMENT OF LIMIT VALUES

b) - IT IS NOT ENOUGH TO PROVE

$\nabla\cdot\vec{F} = 0$ FOR $\vec{F}$ TO BE CONSERVATIVE

THIS IS ONLY A NECESSARY CONDITION

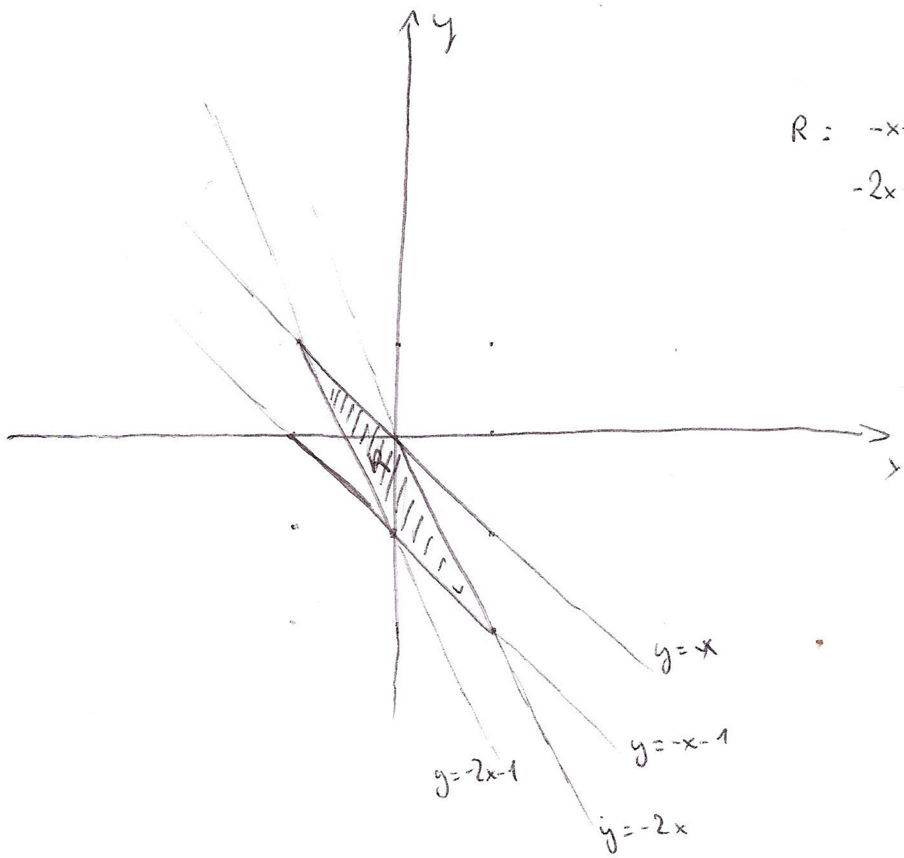
TO HAVE A PROOF YOU NEED EITHER OBSERVE • F IS DEFINED OVER $\mathbb{R}^3$ WHICH IS SIMPLY CONNECTED OR COMPUTE POTENTIAL

- WHEN YOU COMPUTE POTENTIAL YOU HAVE IMPLICATIONS IN ONE DIRECTION SO IF YOU WANT TO USE φ AS POTENTIAL YOU NEED AT THE END TO CHECK THAT $\nabla\varphi = \vec{F}$

c) THE INTEGRAL COMPUTED BY DEFINITION IS VERY HARD TO DO, MAYBE EVEN IMPOSSIBLE SO I DIDN'T RESPECT SUCH AN APPROACH.

EX.2.

a) R IS BOUNDED BY $y=-x$, $y=-x-1$, $y=-2x$, $y=-2x-1$



$R: \; -x-1 \leq y \leq -x$
$\quad -2x-1 \leq y \leq -2x$

$x=u-v$, $y=-2u+v$ $\Rightarrow$ $x+y=-u$, $y+2x=-v$

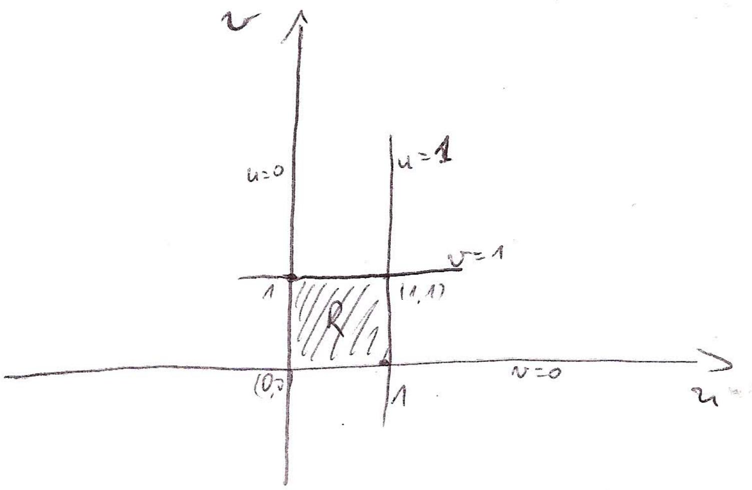
INEQUALITIES DESCRIBING R IN u,v:

$y \leq -x \iff y+x \leq 0 \iff -u \leq 0 \iff u \geq 0$

$y \leq -2x \iff y+2x \leq 0 \iff -v \leq 0 \iff v \geq 0$

$y \geq -x-1 \iff y+x \geq -1 \iff -u \geq -1 \iff u \leq 1$

$y \geq -2x-1 \iff y+2x \geq -1 \iff -v \geq -1 \iff v \leq 1$



b)

$$\iint_{R(x,y)} (y+2x)(x+y-1)\, dA = \iint_{R(u,v)} (-v)(-u+1) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv =$$

$$= \int_0^1 \int_0^1 v(u-1) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = (*)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = |-1| = 1$$

$$* = \int_0^1 \int_0^1 v(u-1) \cdot 1\, du\, dv = \int_0^1 v\, dv \cdot \int_0^1 (u-1)\, du = \frac{1}{2}\left(\frac{1}{2} - 1\right) = -\frac{1}{4}$$

COMMENTS ON WHAT I OBSERVED IN YOUR SOLUTIONS:

a) SOME PEOPLE HAD PROBLEMS DRAWING FIRST PICTURE

FOR SECOND PICTURE I WANTED TO SEE THE COMPUTATION

b) THE FORMULA WITH THE JACOBIAN INVOLVES THE <u>ABSOLUTE VALUE</u> OF THE JACOBIAN NOT THE JACOBIAN ITSELF, I CUT POINTS HERE FOR NOT OBSERVING THAT

I WAS VERY STRICT WHEN SOMEONE DIDN'T USE THE JACOBIAN AT ALL

(NOTE THAT IN THIS CASE THIS GAVE THE SAME RESULT)

Ex. 3. $\vec{F}(x,y,z) = (y+z)\vec{i} - (z+6y^2)\vec{j} + (z\cos y + y\sin x)\vec{k}$

a)

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 - 12y + \cos y$$

$$\operatorname{CURL}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$= \left(-z\sin y + \sin x + 1\right)\vec{i} - \left(y\cos x - 1\right)\vec{j} + (-1)\vec{k}$$

b) C : $x^2+y^2+z^2=5$, $z=1$

I.E. $x^2+y^2+1=5$, $z=1$ $\Rightarrow$ $x^2+y^2=4$, $z=1$

I.E. C IS A CIRCLE OF RADIUS 2 ON THE PLANE $z=1$

LET D BE THE DISC OF RADIUS 2 BOUNDED BY THIS CIRCLE C ON THE SAME PLANE $z=1$

NOTE THAT D IS ORIENTED BY $\vec{k}$ (CONSTANT UNIT VECTOR FIELD DEFINED ON D) AND THIS ORIENTATION INDUCES THE ORIENTATION OF C THAT WAS CHOSEN (ANTICLOCKWISE LOOKING FROM ABOVE)

BY STOKES THM:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \operatorname{CURL}(\vec{F})\cdot\vec{k}\,dS = \iint_D (-1)\,dS = -\text{ AREA OF } D \underset{\substack{\uparrow \\ \text{D CIRCLE} \\ \text{OF RADIUS 2}}}{=} -4\pi$$

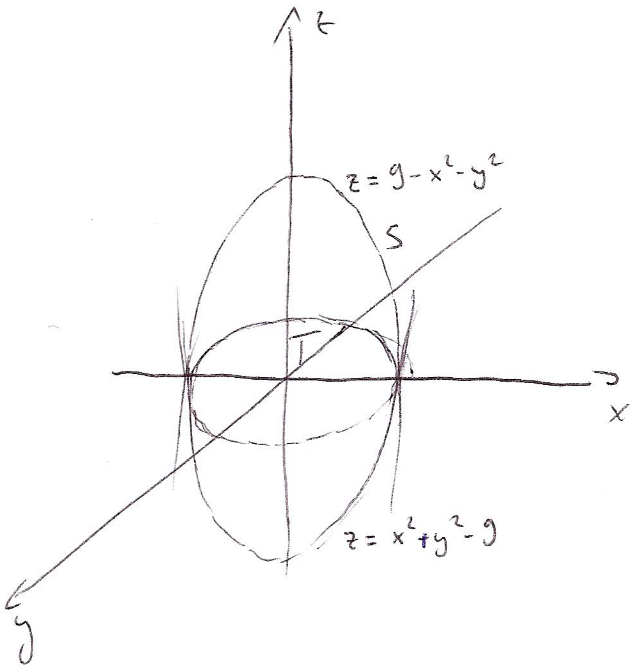
COMMENTS ON YOUR SOLUTIONS:

a) I DIDN'T TAKE INTO ACCOUNT SMALL MISTAKES IN COMPUTATION BUT THERE SHOULD BE NO VECTORS IN DIV AND ONLY VECTORS IN CURL.

b) AN ARGUMENT ABOUT THE RIGHT ORIENTATION ($\vec{k}$ NOT $-\vec{k}$) SHOULD BE GIVEN. ALSO A DESCRIPTION OF THE SURFACE ON WHICH WE USE STOKES THM SHOULD BE GIVEN (I.E. D THE DISC, NOT PART OF SPHERE b cont.) SOME PEOPLE MANAGED TO GO THROUGH THE INTEGRAL BY DIRECT COMPUTATION FROM DEFINITION, THAT IS OK BUT PROVIDED THERE ARE NO OBVIOUS MISTAKES IN THE INTEGRALS.

Ex. 4.

a)



T IS BOUNDED BY $\quad z = 9 - x^2 - y^2$ AND $z = x^2 + y^2 - 9$
$z \geq 0$

I.E. $\quad T: \quad x^2 + y^2 - 9 \leq z \leq 9 - x^2 - y^2$

IN PARTICULAR $\quad x^2 + y^2 - 9 \leq 9 - x^2 - y^2 \quad$ ON T

I.E. $\quad x^2 + y^2 \leq 9$

IN POLAR COORDINATES z, r, θ WITH $\quad x = r\cos\theta, \; y = r\sin\theta, \; z = z$

WE HAVE $\quad T: \quad r^2 - 9 \leq z \leq 9 - r^2 \quad$ AND IN PARTICULAR $r^2 \leq 9$

I.E $\quad 0 \leq r \leq 3$

THERE IS NO CONDITION ON θ.
I.E $\quad \theta \in [0, 2\pi]$

BY CHANGE OF COORDINATES AND ITERATION WE HAVE :

$$\iiint_T z + 1 \, dV = \iiint_T z \, dV + \iiint_1 dV = \underbrace{\iiint_T z \, dv}_{=0} + \int_0^{2\pi}\int_0^3\int_{r^2-9}^{9-r^2} r \, dz \, dr \, d\theta$$

BECAUSE z IS ANTI SYMMETRIC VIA $(x,y,z) \mapsto (x,y,-z)$ AND T IS SYMMETRIC W.R.T THIS TRANSFORMATION

$$= \underbrace{\int_0^{2\pi}\int_0^3 2r(9-r^2) \, dr \, d\theta}_{=(**)} = 2\pi\left(\frac{18r^2}{2} - 2\frac{r^4}{4}\right)\Bigg|_0^3 = \underline{81\pi}$$

b) $\vec{F}(x,y,z) = 2yz^2\,\vec{i} - 2xz^2\,\vec{j} + z\,\vec{k}$

$$\iint_S \vec{F}\cdot\hat{N}\,dS = \iint_{\pi(S)} \vec{F}\cdot\frac{\nabla g}{|\nabla g\cdot\vec{P}|}\,dA = (*)$$

$\vec{P} = \vec{k}$

WHERE $\pi : S \to$ PLANE (x,y)
1:1 PROJECTION
$\pi(S)$ IS THE DISC
$x^2+y^2 \le 9$

g IS THE FUNCTION
$9 - x^2 + y^2 + z = g$
THE g GROWS ~~OUT~~ OUT OF T
i.e. ∇g IS NORMAL OUT OF T

$\nabla g = -2x\vec{i} - 2y\vec{j} + \vec{k}$

$|\nabla g\cdot\vec{k}| = |1| = 1$

$\vec{F}\cdot\nabla g = -4xyz^2 + 4xyz^2 + z = z$

POLAR COORDINATES

SEE PREVIOUS PAGE

$$(*) = \iint_{x^2+y^2\le 9} z\,dA = \iint_{x^2+y^2\le 9} 9 - x^2 - y^2\,dA = \int_0^{2\pi}\int_0^{3} (9-r^2)\,r\,dr\,d\theta = \frac{(**)}{2} = \frac{81\pi}{2}$$

COMMENTS: a) SOME OF YOU USED THE FORMULA FOR VOL OF PARABOLOID THAT WAS ALSO OK.

b) THE ARGUMENT TELLING WHY WE USE g NOT $-g$ IS NEEDED (ORIENTATION)

a) THE WRONG LIMITS $z \in [0, 9 - r^2]$ TAKEN OUT OF THE HAT ARE CONSIDERED COMPLETELY WRONG. AN IMPORTANT PART OF THIS EXERCISE IS TO SET LIMITS CORRECTLY.

b) THE DIVERGENCE THM CANNOT!!! BE USED THE SURFACE S IS NOT THE BOUNDARY OF ANY DOMAIN, IT IS PART OF THE BOUNDARY. A PRIORI YOU COULD USE DIVERGENCE THM FOR T BUT THEN NEED TO COMPARE THE SURFACE INTEGRAL ON THE LOWER BOUNDARY ~~FRO~~ TO THE ONE THAT WE WANT. I KNOW THAT YOU GET THE SAME RESULT FROM DIVERGENCE THM BUT THE ARGUMENT IS WRONG AND I DO NOT COUNT IT!