THE FACULTY OF SCIENCE AND TECHNOLOGY

EXAM II: MAT300 Vectoranalysis

DATE: 4th March 2016, 09:00 - 13:00

PERMITTED TO USE:
Rottmann: Matematisk formelsamling
Calculators: HP 30S, Casio FX82, TI-30, Citizen SR-270X, Texas BA II Plus, HP17bII

THE EXERCISE SHEET CONTAINS 4 EXERCISES ON 2 PAGES + 1 PAGE WITH FORMULAS

EXERCISE 1

Let C be the curve: $\mathbf{r}(t) = e^t \cos t\ \mathbf{i} - e^t \sin t\ \mathbf{j} + t\ \mathbf{k}; \quad 0 \le t \le \pi.$

a) Compute the line integral

$$\int_C x^2 + y^2 + e^{2z}\ ds.$$

b) Prove that the vector field $\mathbf{F}(x, y, z) = (2x+y+yz)\mathbf{i}+(e^z+x+xz)\mathbf{j}+(ye^z+xy+3z^2)\mathbf{k}$

is conservative.

c) Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

EXERCISE 2

Consider the transformation $x = u + 3v; \quad y = 2u + v$, between
(u, v) -coordinates and (x, y) -coordinates.
Let R be the bounded domain on the plane xy bounded by the lines:
$y = 2x,\ y = 2x - 5,\ 3y = x,$ end $3y = x + 5.$

a) Draw R in the plane xy and the image of R in the plane uv.

b) Compute the double integral

$$\iint_R (y - 2x)e^{6y-2x+1} dA,$$

using the above change of coordinates.

EXERCISE 3

Let F be the vector field $\mathbf{F}(x, y, z) = (y + ze^x)\mathbf{i} - (z\cos^2 x + 6y^2)\mathbf{j} + (z\cos x + y\sin^2 x)\mathbf{k}$.

a) Compute *div* **F** and *curl* **F**.

b) Let C be the curve of intersection of the sphere $x^2 + y^2 + z^2 = 5$ with the plane $x = 1$. Let C be oriented counterclockwise looking from $(0, 0, 0)$. Compute the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

EXERCISE 4

Let S be the part of the sphere $6 = x^2 + y^2 + z^2$, that lies above the plane xy.
Let T be the domain bounded by the surface S and the paraboloid $z = x^2 + y^2$.

a) Compute

$$\iiint_T (z + 1)dV.$$

b) Consider the vector field $\mathbf{F}(x, y, z) = (xz + y)\mathbf{i} + (yz - x)\mathbf{j} + (z^2 + z)\mathbf{k}$.
Compute the flux

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

of $\mathbf{F}$ in the direction out of the ball $x^2 + y^2 + z^2 \leq 6$, where $\hat{\mathbf{N}}$ is the unit normal vector to S.

Good luck!

Formulas:

Line integral of a function f along a curve C: $\mathbf{r} = \mathbf{r}(t), \quad a \le t \le b$:

$$\int_C f ds = \int_a^b f(r(t))|\frac{d\mathbf{r}}{dt}|\ dt.$$

Line integral of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, along a curve C: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 d_y + F_3 dz = \int_a^b (F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt})dt.$$

Surface integral of a function f over a surface $S:\ g(x, y, z) = K$ (K is a constant):

$$\iint_S f\ dS = \iint_R f \frac{|\nabla g|}{|\nabla g \cdot \mathbf{P}|}\ dA.$$

Stokes' theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\ dS = \iint_S (\nabla \times \mathbf{F}) \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{P}|}\ dA.$$

Divergence theorem:

$$\iiint_D \nabla \cdot \mathbf{F}\ dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{N}}\ dS.$$

$div\ \mathbf{F} = \nabla \cdot \mathbf{F}, \quad curl\ \mathbf{F} = \nabla \times \mathbf{F}, \quad \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

Cylindrical coordinates: $(r\cos\theta, r\sin\theta, z) = (x, y, z)$.
Spherical coordinates $(R\sin\phi\cos\theta, R\sin\phi\sin\theta, Rcos\phi) = (x, y, z)$

EXAM II

1) a) $\vec{r}(t) = e^t \cos t\, \vec{i} - e^t \sin t\, \vec{j} + t\, \vec{k} \qquad 0 \le t \le \pi$

$$\frac{d\vec{r}}{dt} = \left(e^t(-\sin t) + e^t \cos t\right)\vec{i} - \left(e^t \cos t + e^t \sin t\right)\vec{j} + \vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(e^t(\cos t - \sin t)\right)^2 + \left(e^t(\cos t + \sin t)\right)^2 + 1} =$$

$$= \sqrt{e^{2t}(\cos^2 t + \sin^2 t - 2\sin t\cos t) + e^{2t}(\cos^2 t + \sin^2 t + 2\sin t\cos t) + 1}$$

$$= \sqrt{2e^{2t} + 1}$$

$$\int_C x^2 + y^2 + e^{2z}\, ds = \int_0^{\pi} \left(e^{2t}\cos^2 t + e^{2t}\sin^2 t + e^{2t}\right)\sqrt{2e^{2t}+1}\, dt$$

$$= \int_0^{\pi} 2e^{2t}\sqrt{2e^{2t}+1}\, dt \qquad u = 2e^{2t} + 1 \qquad \frac{du}{dt} = 4e^{2t}$$

$$\int 2e^{2t}\sqrt{2e^{2t}+1}\, dt = \int \frac{\sqrt{u}}{2}\, du = \frac{1}{2}\,\frac{2}{3}\, u^{\frac{3}{2}} + C = \frac{1}{3} u^{\frac{3}{2}} + C =$$

$$= \frac{1}{3}\left(2e^{2t}+1\right)^{\frac{3}{2}} + C$$

$$\int_0^{\pi} 2e^{2t}\sqrt{2e^{2t}+1} = \frac{1}{3}\left(2e^{2\pi}+1\right)^{\frac{3}{2}} - \frac{1}{3}\, 3^{\frac{3}{2}}$$

b) $\vec{F}(x,y,z) = (2x + y + yz)\,\vec{i} + (e^z + x + xz)\,\vec{j} + (ye^z + xy + 3z^2)\,\vec{k}$

WE LOOK FOR POTENTIAL φ SUCH THAT

$$\frac{\partial \varphi}{\partial x} = 2x+y+yz \quad , \quad \frac{\partial \varphi}{\partial y} = e^z + x + xz \quad , \quad \frac{\partial \varphi}{\partial z} = ye^z + xy + 3z^2$$

$$\varphi = \int 2x+y+yz \, dx = x^2 + xy + xyz + C_1(y,z)$$

$$\frac{\partial \varphi}{\partial y} = x + xz + \frac{\partial C_1}{\partial y} = x + xz + e^z \Rightarrow \frac{\partial C_1}{\partial y} = e^z \Rightarrow C_1(y,z) = \int e^z dy = ye^z + C_2(z)$$

$$\frac{\partial \varphi}{\partial z} = xy + ye^z + C_2'(z) = xy + ye^z + 3z^2 \Rightarrow C_2(z) = \int 3z^2 dz = z^3 + C_3$$

$$\varphi = x^2 + xy + xyz + ye^z + z^3 + \underset{\substack{\uparrow \\ \text{CONSTANT}}}{C_3}$$

CHECK

$$\frac{\partial \varphi}{\partial x} = 2x+y+yz \checkmark \quad , \quad \frac{\partial \varphi}{\partial y} = x + xz + e^z \checkmark \quad , \quad \frac{\partial \varphi}{\partial z} = xy + ye^z + 3z^2 \checkmark$$

ok

SO φ - POTENTIAL SO $\vec{F}$ CONSERVATIVE

c)

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\vec{r}(0)}^{\vec{r}(\pi)} \vec{F} \cdot d\vec{r} = \varphi(\vec{r}(\pi)) - \varphi(\vec{r}(0)) =$$

$$= \varphi((e^{\pi}\cos\pi, -e^{\pi}\sin\pi, \pi)) - \varphi((e^0\cos 0, e^0 \sin 0, 0))$$

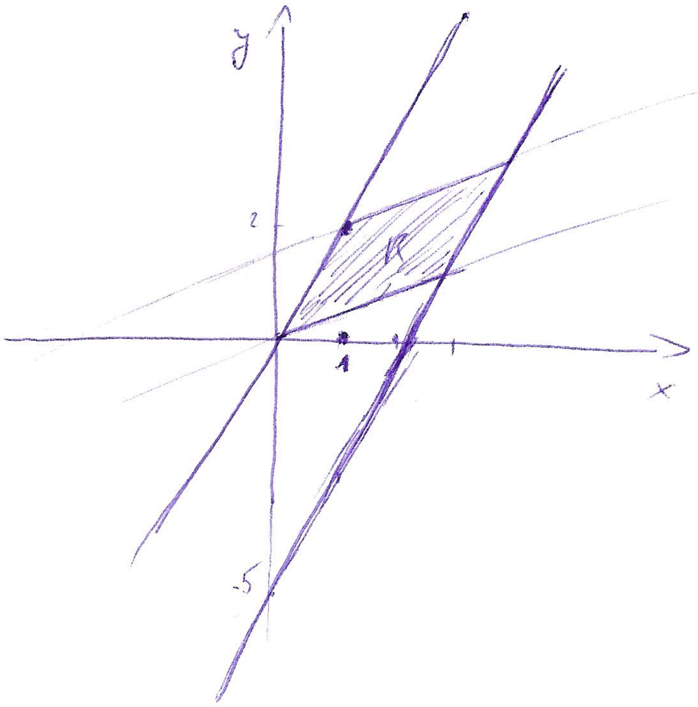
$$= \varphi((-e^{\pi}, 0, \pi)) - \varphi((1, 0, 0)) =$$

$$= e^{2\pi} + \pi^3 - 1$$

2. $x = u + 3v \qquad y = 2u + v$

R BOUNDED BY $\quad y = 2x, \; y = 2x - 5, \; 3y = x, \; 3y = x + 5$

a)

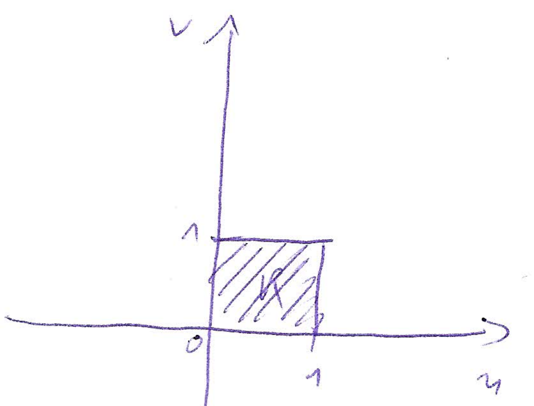


R:

$$2x - 5 \le y \le 2x$$
$$x \le 3y \le x + 5$$

$$\begin{cases} x = u + 3v \\ y = 2u + v \end{cases} \Rightarrow \begin{cases} y - 2x = -5v \\ 3y - x = 5u \end{cases}$$

R: $\quad \begin{matrix} -5 \le y - 2x \le 0 \\ 0 \le 3y - x \le 5 \end{matrix} \iff \begin{matrix} -5 \le -5v \le 0 \\ 0 \le 5u \le 5 \end{matrix} \iff \begin{matrix} 0 \le v \le 1 \\ 0 \le u \le 1 \end{matrix}$



b)

$$(*) = \iint_R (y-2x)\, e^{6y-2x+1}\, dA = \int_0^1\int_0^1 (-5v)\, e^{10u+1} \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\right| = |1-6| = |-5| = 5$$

$$(*) = \int_0^1\int_0^1 (-5v)\, e^{10u+1} \cdot 5\, du\, dv = -25\left(\int_0^1 v\, dv\right)\left(\int_0^1 e^{10u+1}\, du\right) =$$

$$= -25\left(\frac{1}{2}v^2\Big|_0^1\right)\left(\frac{1}{10}e^{10u+1}\Big|_0^1\right) = -25 \cdot \frac{1}{2}\left(\frac{e^{11}}{10} - \frac{e}{10}\right) =$$

$$= -\frac{25}{20}(e^{11} - e)$$

3. $\vec{F}(x,y,z) = (y + ze^{x})\vec{i} - (z\cos^2 x + 6y^2)\vec{j} + (z\cos x + y\sin^2 x)\vec{k}$

$$\operatorname{div}\vec{F} = ze^{x} + 12y + \cos x$$

~~$y+ze^x \quad -(z\cos^2 x + 6y^2) \quad z\cos x + y\sin^2 x$~~

~~$\operatorname{curl}\vec{F} = \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$~~

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+ze^{x} & -(z\cos^2 x+6y^2) & z\cos x + y\sin^2 x \end{vmatrix} = (\sin^2 x + \cos^2 x)\vec{i} -$$

$$(-z\sin x + y\, 2\sin x\cos x - e^{x})\vec{j} +$$

$$(z\, 2\cos x\sin x + 1)\,\vec{k}$$

b) $\begin{cases} x^2+y^2+z^2=5 \\ x=1 \end{cases} \Rightarrow \begin{cases} y^2+z^2=4 \\ x=1 \end{cases}$ C circle with radius 2 on plane $x=1$

C - closed curve $\quad C = \partial S \quad$ S - DISC OF RADIUS 2 ON PLANE $x=1$

$S: \begin{cases} y^2+z^2 \leq 4 \\ x=1 \end{cases}$

STOKES THM:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{N}\, dS$$

$\hat{N}$ - NORMAL TO S SUCH THAT WHEN WALKING ALONG C ~~STANDING~~ COUNTER CLOCKWISE LOOKING FROM $(0,0,0)$ YOU HAVE S ON YOUR LEFT

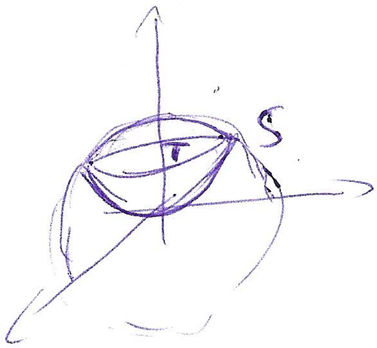
I.E. $\hat{N} = -\vec{i}$

$$\iint_S (\text{curl}\,\vec{F}) \cdot (-\vec{i})\, dS = \iint_S (\sin^2 x + \cos^2 x) \cdot (-1)\, dS = -\iint_S dS =$$

$- \text{AREA OF } S = -4\pi$

S IS DISC OF RADIUS 2

4.



$$T: \quad x^2+y^2+z^2 \leq 6$$
$$x^2+y^2 \leq z$$

$$\Longleftrightarrow \quad z^2 \leq 6 - x^2 - y^2$$
$$0 \leq x^2+y^2 \leq z$$
$$\Longleftrightarrow \quad x^2+y^2 \leq z \leq \sqrt{6-x^2-y^2}$$

$$\iiint_T (z+1)\, dV = ?$$

IN CYLINDRICAL COORDINATES

$$x^2+y^2 = r^2$$

$$r^2 \leq z \leq \sqrt{6-r^2}$$

IN PARTICULAR LIMITS FOR r:

$$r \geq 0, \quad \sqrt{6-r^2} \geq r^2 \Rightarrow$$
$$r^4 \geq 6 - r^2 \geq 0$$

$$\Rightarrow \quad 0 \leq r \leq \sqrt{2}$$

$$\iiint_T (z+1)\, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} (z+1)\, r\, dz\, dr\, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r \cdot \left(\frac{1}{2} z^2 + z\right)\Big|_{r^2}^{\sqrt{6-r^2}} dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} \frac{r}{2}(6-r^2) + r\sqrt{6-r^2} - \left(\frac{r^5}{2} + r^3\right) dr\, d\theta$$

$$= 2\pi \left(\int_0^{\sqrt{2}} -\frac{r^5}{2} - \frac{3}{2} r^3 + 3r\, dr + \int_0^{\sqrt{2}} r\sqrt{6-r^2}\, dr \right) =$$

$$u = 6 - r^2$$
$$du = -2r\, dr$$

$$= 2\pi \left(\left(-\frac{1}{12} r^6 - \frac{3}{8} r^4 + \frac{3}{2} r^2\right)\Big|_0^{\sqrt{2}} + \left(\left(-\frac{1}{2}\right)(6-r^2)^{\frac{3}{2}} \cdot \frac{2}{3}\right)\Big|_0^{\sqrt{2}} \right) =$$

$$= 2\pi\left(-\frac{8}{12}-\frac{12}{8}+3+\left(-\frac{1}{3}\left(2^{\frac{3}{2}}-6^{\frac{3}{2}}\right)\right)\right) =$$

$$\frac{3}{2}\pi + 4\pi\sqrt{6} - \frac{4}{3}\pi\sqrt{2}$$

b) $\vec{F}(x,y,z) = (xz+y)\vec{i} + (yz-x)\vec{j} + (z^2+z)\vec{k}$

S: $x^2+y^2+z^2=6$, $z>0$

$g = x^2+y^2+z^2$ $\qquad \nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

S _ LEVEL SURFACE FOR g

$$\vec{F}\cdot\nabla g = 2x^2z + 2xy + 2y^2z - 2xy + 2z^3 + 2z^2 \overset{\text{ON } S}{=} 12z + 2z^2$$

S HAS 1:1 PROJECTION ONTO DISC D: $x^2+y^2\le 6$ ON PLANE xy WITH NORMAL VECTOR $\vec{k}$

$$|\nabla g \cdot k| = 2z$$

$$\iint_S \vec{F}\cdot\hat{N}\,dS = \iint_D \frac{12z+2z^2}{2z}\,dA = \iint_D (1+z)\,dA$$

POLAR COORDINATES
$z = \sqrt{6-r^2}$ ON S

$$= \int_0^{2\pi}\int_0^{\sqrt{6}} \left(1+\sqrt{6-r^2}\right) r\,dr\,d\theta = 2\pi\cdot\left(\frac{1}{2}r^2\Big|_0^{\sqrt{6}} + \left(-\frac{1}{3}\right)(6-r^2)^{\frac{3}{2}}\Big|_0^{\sqrt{6}}\right)$$

$$= 6\pi + 4\pi\sqrt{6}$$