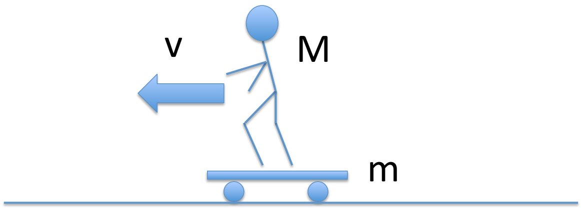
Høsten 2015

FYS100 Fysikk
Eksamen/Exam
Solutions

Problem 1: Skate-bored (15 points)



A child of mass $M = 45.0$ kg is on a skateboard of mass $m = 3.00$ kg travelling at speed $v_i = 4.00$ m/s down a street. The child now somehow manages to jump straight up in the air, relative to the street.

a) *What is the final speed of the skateboard, relative to the street?*

Solution: This is clearly a momentum conservation problem. He jumps, and so energy is not conserved (it comes from his muscles), and the normal force provides impulse upwards. But in the horizontal direction, there is momentum conservation (unless we would start worrying about details of the rolling motion of the wheels; there is no mention of the mass of radius of the wheels, so this is clearly not the intention).

Momentum in the horizontal direction is therefore conserved. That means that with v_i being the initial speed of both skate-board and child relatives to the ground; and v_f the final speed of the skate-board (the final speed of the child relative tot he ground is 0), we have:

$$(M+m)v_i = mv_f \to v_f = \frac{M+m}{m}v_i = 64 \text{ m/s} \tag{1}$$

Some of you solved this by going to the frame v_i. That's of course fine.

b) *What is the final kinetic energy of the skate-board divided by the total initial kinetic energy?*

Solution: Given what v_f it is just to insert in the expression for kinetic energy:

$$\frac{E_f^s}{E_i^s + E_i^c} = \frac{mv_f^2/2}{mv_i^2/2 + Mv_i^2/2} = \frac{m}{m+M}\left(\frac{v_f}{v_i}\right)^2 = \frac{M+m}{m} = 16. \tag{2}$$

c) What if he had jumped straight up in the air, not relative to the ground, but relative to the skateboard?

Solution: Think of the frame where the skateboard is initially at rest (the frame of v_i). Then he just jumps straight up from a skate-board at rest. They don't move relative to each other, so in particular, the skate-board doesn't change speed. It also doesn't change energy. $v_i = v_f$,

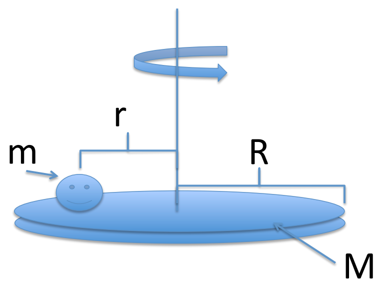
$$\frac{E_f^s}{E_i^s + E_i^c} = \frac{m}{m+M} = \frac{1}{16}. \tag{3}$$

For all questions, provide an algebraic expression, as well as the numerical result.

Typical mistakes: Not using momentum conservation. Using energy conservation. Using rocket equation (wrongly). Providing a solution in b) with the unit J (it's a ratio of energies, unit cancels out). Calculating the energy ratio and getting the result 0. Not making a sketch as required.

Partial credit given for: mentioning momentum conservation but doing it wrong. Writing down the energy ratio, but getting the wrong result. Using energy conservation (which is wrong) but at least calculating that correctly (to the wrong number).

Problem 2: Disc-o-frog (15 points)



A frog of mass $m = 0.100$ kg is sitting on a rotating disc of mass $M = 0.500$ kg and radius $R = 15.0$ cm. The frog is at a distance $r = 12.0$ cm from the centre of the disc (which is also the axis of rotation), and the disc is rotating without friction with a period of $T = 3.00$ s. The frog now jumps straight up in the air, relative to the ground. The moment of inertia of a disc is $I_d = MR^2/2$.

a) *What is the angular speed of the disc, just after the frog has jumped?*

Solution: This problem is completely analogous to problem 1, except that it is now angular momentum that is conserved (not energy; the frog jumps and provides energy). First convert from the period to the angular speed

$$\omega_i = \frac{2\pi}{T} \tag{4}$$

and writing down angular momentum conservation (the frog has by assumption zero final angular speed)

$$(I_{frog} + I_d)\omega_i = I_d\omega_f \to \omega_f = \frac{I_d + I_{frog}}{I_d}\omega_i \tag{5}$$

we need to find ω_f for the disc. Using $I_d = MR^2/2$ (disc) and $I_{frog} = mr^2$ (point particle; nowhere any mention of the moment of inertia of a frog!), we find

$$\omega_f = \left[1 + \frac{2m}{M}\frac{r^2}{R^2}\right]\omega_i = 2.63 \text{ rad/s} \tag{6}$$

b) *What is the final kinetic energy of the disc divided by the total initial kinetic energy?*

Solution: Again analogous to problem 1 b, we simply take the energy expressions for a rigid body, and write

$$\frac{E_f^d}{E_i^d + E_i^{frog}} = \frac{I_d\omega_f^2/2}{I_d\omega_i^2/2 + I_{frog}\omega_i^2/2} = \frac{I_d}{I_d + I_{frog}}\left(\frac{\omega_f}{\omega_i}\right)^2 = \frac{I_d + I_{frig}}{I_d} = 1 + \frac{2m}{M}\frac{r^2}{R^2} = 1.26 \tag{7}$$

c) *What if the frog had jumped straight up, not relative to the ground, but relative to the instantaneous velocity at the point on the disc, form where it jumped?*

Solution: Think of the frame rotating with the disc. Then he just jumps up, but then flies off the edge. But initially, they don't move relative to each other, so the disc doesn't change speed. It also doesn't change energy. This is again analogous to 1 c. We get $\omega_i = \omega_f$,

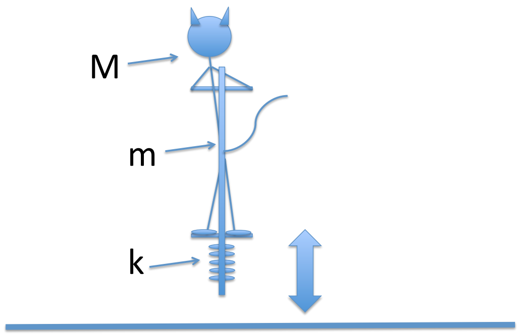
$$\frac{E_f^d}{E_i^d + E_i^{frog}} = \frac{I_d}{I_d + I_{frog}} = \frac{1}{1.26} = 0.796. \tag{8}$$

For all questions, provide an algebraic expression, as well as the numerical result.

Typical mistakes: Not using angular momentum conservation. Using energy conservation. Providing a solution in b) with the unit J (it's a ratio of energies, unit cancels out). Calculating the energy ratio and getting the result 0. Not making a sketch as required.

Partial credit given for: Mentioning angular momentum conservation but doing it wrong. Writing down the energy ratio, but getting the wrong result. Using energy conservation (which is wrong) but at least calculating that correctly (to the wrong number).

Problem 3: Tigger (15 points)



A tiger is jumping up and down on a pogo stick. For the purpose of this problem, we will think of such a stick as being a (massless) spring with a tiger + stick on top. We will assume that no friction is involved. At his highest point, the bottom of the (unstretched/unsqueezed) spring is at a height of $h = 40.0$ *cm from the ground. The spring constant of the pogo stick is taken to be* $k = 50000$ *N/m. The mass of the tiger is* $M = 70.0$ *kg and of the pogo stick* $m = 5.00$ *kg.*

a) *What is the total energy of the system? (For the purpose for the gravitational potential energy, this is taken to be relative to the lowest point of the jumping motion, when the spring is on the ground, and the spring is squeezed; think carefully about this.)*

Solution: This problem appeared as a hand-in problem this year, and involves energy conservation. There is kinetic energy, gravitational potential energy and spring potential energy. When at the lowest point, all the energy is in spring potential energy,

$$E_{\text{total}} = \frac{1}{2} k dx^2. \tag{9}$$

. dx is the squeezing amount, and we can call d the height from the ground (the length of the spring, when squeezed). We will never need to know d, but just keep in mind that when the spring is un-squeezed, it is $d + dx$ long.
But what is dx? At the highest point, the block CM has moved up $h + dx$ (since the total height is the squeezed spring d + the squeezing $dx + h$). And there all the energy is in gravitational potential energy

$$E_{\text{total}} = (M + m)g(h + dx). \tag{10}$$

Energy is conserved, and putting these two equal, we find

$$\frac{k}{2} dx^2 - (M + m)g(dx + h) = 0 \rightarrow dx = 0.124 \text{ m} \tag{11}$$

There is a second, negative solution, which we discard. Then the total energy is

$$E_{\text{total}} = (M+m)g(dx+h) = 385 \text{ J}. \tag{12}$$

b) *Compute the value of the different energy components at the following points: At the highest point of the jumping motion (A); At the lowest point of the jumping motion (B); at the point where the spring of the pogo stick is at its equilibrium point (C).*

Solution:
At A all is in the gravitational potential energy:
$E_g = 385$ J, $E_s = 0$, $E_{kin} = 0$.

At B all is in the spring potential energy
$E_g = 0$, $E_s = 385$ J, $E_{kin} = 0$.

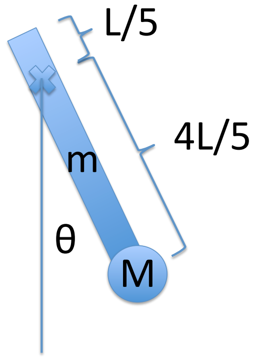
At C, the block has moved dx up, so
$E_g = (M+m)gdx = 91.2$ J, $E_s = 0$, $E_{kin} = E_{\text{total}} - E_g = 294$ J.

For all questions, provide an algebraic expression, as well as the numerical result.

Typical mistakes: Not making a sketch as required. Forgetting about d. Forgetting about dx is the change of height. Not solving for x first. Solving wrongly for dx. Calculating the energies wrong.

Partial credit given for: Energy conservation but doing it wrong. Writing down the energies for A, B, C but with the wrong numbers (because dx was ignored/calculated wrong).

Problem 4: A second pendulum (20+10 points)



A pendulum consists of a uniform stick of length $L = 1.50$ m and mass $m = 1.00$ kg, on which is attached a block of mass $M = 2.00$ kg. The axis of the pendulum is at a distance of $L/5$ from the top end. The mass is attached at the bottom of the stick (L from the top end, $(4/5)L$ from the axis). The moment of inertia of the stick around its centre of mass is $(1/12)mL^2$, and the block may be treated as a point particle, for the purpose of its moment of inertia.

a) *What is the moment of inertia of the whole pendulum around the axis?*

Solution: The moment of inertia has two components, to be summed. From the point-like block:

$$I_b = M\left(\frac{4}{5}L\right)^2 \tag{13}$$

And for the stick

$$I_s = \frac{1}{12}mL^2 + m\left(\frac{L}{2} - \frac{L}{5}\right)^2 \tag{14}$$

which arises from using the parallel axis theorem. Summing them up, we get

$$I = I_s + I_b = \left(\frac{16}{25}M + \frac{13}{75}m\right)L^2 = 3.27 \text{ kg m}^2. \tag{15}$$

b) *Where is the centre of mass of the stick-block object, and what is its distance from the axis?*

Solution: It is again a sum of contributions: The stick has CM at $L/2$ with mass m; and the block at L with mass M (relative to the end of the stick). We have

$$x_{CM} = \frac{ML + m\frac{L}{2}}{M+m} = \frac{5}{6}L. \tag{16}$$

The distance from the axis is then

$$d = \left(\frac{5}{6} - \frac{1}{5}\right)L = \frac{19}{30}L = 0.95 \text{ m}. \tag{17}$$

c) What is the period T of oscillation of this physical pendulum, assuming the small-angle approximation ($\sin\theta \simeq \theta$)?

Solution: The gravitational force acts at the centre of mass, with an arm d and with an orthogonal component of $(M+m)g\sin\theta$. The torque around the axis is therefore

$$|\tau| = d(M+m)g\sin\theta. \tag{18}$$

It's opposite to the angle growing, so clockwise in the picture. We write the angular version of Newton's second law, and use the small-angle approximation

$$I\alpha = dg(M+m)\sin\theta \simeq dg(M+m)\theta \tag{19}$$

Comparing this to the harmonic oscillator equation for the angle

$$\alpha = \frac{d^2\theta}{dt^2} = -\omega^2\theta, \tag{20}$$

we find

$$\to \omega^2 = \frac{d(M+m)g}{I} \tag{21}$$

Inserting I and d, we find for the period $T = 2\pi/\omega$

$$T = 2\pi\sqrt{\frac{(\frac{16}{25}M + \frac{13}{75}m)L^2}{(M+m)g19/30L}} = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{\frac{16}{25}M + \frac{13}{75}m}{(M+m)19/30}} = 2.15 \text{ s.} \tag{22}$$

d) *Where should one instead put the axis, for the pendulum to have a period of* 2.00 *s?*

Solution: This was for extra 10 points: The point is to instead of using $L/5$ as the position of the axis, to use the fraction x, so $L/5 \to xL$. CM is the same, but d changes to $(5/6 - x)L$. Moment of inertia changes to

$$I = I_s + I_b = \left(M(1-x)^2 + \frac{1}{12}m + m\left(\frac{1}{2} - x\right)^2\right)L^2. \tag{23}$$

Inserting this, we find

$$T = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{(1-x)^2M + ((1/2-x)^2 + 1/12)m)}{(M+m)(5/6-x)}} \tag{24}$$

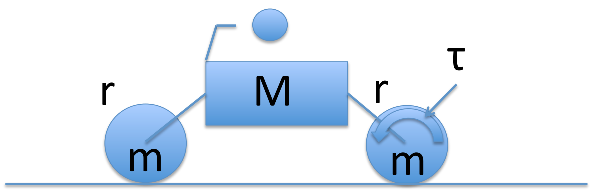
There are wo solutions to this quadratic equation: $x = 0.340$ m and $x = 0.664$ m. Both are correct. But this is a bit of a mess to solve, so the 10 points are in addition tot he full score of 100 points. Some people tried, but nobody completed it.

For all questions, provide an algebraic expression, as well as the numerical result.

Typical mistakes: Not making a sketch as required. Forgetting to include the moment of inertia of the block. Forgetting to move the axis with the parallel axis theorem (so just using the moment of inertia around the CM). Calculating wrong. Calculating the CM wrong. Not knowing what a pendulum is. Not knowing about the oscillator equation. Using an ideal rather than a physical pendulum (see below).

Partial credit given for: Using parallel axis theorem. Including both contributions tot he moment of inertia. Writing down the expression for the angular speed for a physical pendulum. And even just for an ideal pendulum. Having a go at part d), whether successful or not.

Problem 5: Bicycle theory (35 points)



A guy is on a bicycle, which for the purpose of this problem, we will think of as two equal-sized wheels of mass $m = 2.00$ kg and radius $r = 30.0$ cm, connected by a block of mass $M = 80.0$ kg (which includes the guy). The middle block is attached to the centre of each wheel, which is also the centre of mass and axis of rotation of each wheel. The wheels have a moment of inertia around their centre of mass, as if they were thin rings, $I = mr^2$. In some complicated way, pedals provide a torque τ to the back wheel only. There is static friction of the wheels with the ground, and the wheels are rolling without slipping. Ignore the effect of rolling friction and air resistance.

a) *At first, the bike goes along a horizontal road. What is the torque required to accelerate with a rate $a = 2.00$ m/s^2? Hint: Split the system into three sub-systems (front wheel, back wheel, middle block). Keep in mind that the two wheels have very different forces and torques acting on them.*

Solution: For this question, gravity and normal forces are assumed to cancel. Only forces in the horizontal direction are relevant.

System 1: The front wheel is pushed at the CM by the block. Force F_1. It also has a backwards static friction force providing acceleration and torque with arm r, counterclockwise.

System 2: The back wheel has a reaction force from the block, backwards of size F_2, acting at the CM. A friction force forwards giving (clockwise) torque and acceleration. And a torque τ giving counterclockwise torque.

System 3: The block is pushed forwards from behind by F_2; and has a reaction force to system 1) of F_1 backwards. No torque.

We can then write, using the moment of inertia of the wheels $I = mr^2$

$$
\begin{aligned}
1 : F_1 - f_1^s &= ma, && (25)\\
1 : f_1^s r = I\alpha \rightarrow f_1^s &= ma, && (26)\\
2 : f_2^s - F_2 &= ma, && (27)\\
2 : \tau - f_2^s r = I\alpha \rightarrow \frac{\tau}{r} - f_2^s &= ma, && (28)\\
3 : F_2 - F_1 &= Ma. && (29)
\end{aligned}
$$

By simply adding all these equations up, most things cancel out, and we get

$$\frac{\tau}{r} = a(4m + M) \rightarrow \tau = ar(4m + M) = 52.8 \text{ Nm}. \qquad (30)$$

b) *What are the friction forces needed for each wheel to continue rolling without sliding?*

Solution: We just extract from a), that

$$
\begin{aligned}
f_1^s &= ma = 4 \text{ N}, && (31)\\
f_2^s &= \tau/r - ma = a(3m + M) = 172 \text{ N}. && (32)
\end{aligned}
$$

c) *What is the minimum required coefficient of static friction between wheels and ground? Hint: All the weight on the front wheel? The back wheel? Something in-between?*

Solution (6): One distributes the weight with M_1 on the front wheel and M_2 on the back wheel. We don't know what M_1 and M_2 are, but we know that

$$M_1 + M_2 = 2m + M. \qquad (33)$$

The friction forces are then in the limiting case where we need the whole static friction force $f_s = \mu_s |n|$,

$$f_1^s = \mu_s M_1 g, \qquad f_2^s = \mu_s M_2 g \qquad (34)$$

$$\rightarrow f_1^s + f_2^s = \mu_s (M_1 + M_2) g = \mu_s (2m + M) g = a(4m + M). \qquad (35)$$

The last equality come from adding up the two equations from b). We then have that

$$\mu_s = \frac{4m + M}{2m + M}\frac{a}{g} = 0.214 \tag{36}$$

is the minimum necessary. If there is more weight on the back wheel, one needs a smaller μ_s there, but a bigger μ_s on the front wheel. And vice versa. We could also have found M_1 and M_2 by noting that the to friction forces provide the whole acceleration of the bike, so that

$$f_2 - f_1 = \mu_s g(M_2 - M_1) = a(2m + M). \tag{37}$$

But this was not required (approximately 2kg (front) and 82 kg (back)).

The guy now goes up a hill, starting at the bottom with a speed $v = 5.00$ m/s.

d) *What torque is required to continue with this speed, if the hill slopes at an angle of $\theta = 15.0°$ with horizontal?*

Solution: This is very similar to question a). The difference is that there are now forces of gravity in the CM of each system (only in the acceleration equations). And that the acceleration is supposed to be zero. We therefore have

$$\begin{aligned}
1: F_1 - f_1^s - mg\sin\theta &= 0, && (38)\\
1: f_1^s r = I\alpha \rightarrow f_1^s &= 0, && (39)\\
2: f_2^s - F_2 - mg\sin\theta &= 0, && (40)\\
2: \tau - f_2^s r = I\alpha \rightarrow \frac{\tau}{r} - f_2^s &= 0, && (41)\\
3: F_2 - F_1 - Mg\sin\theta &= 0. && (42)
\end{aligned}$$

By simply adding all these equations up, most things again cancel out, and we get

$$\frac{\tau}{r} = (2m + M)g\sin\theta \rightarrow \tau = (2m + M)rg\sin\theta = 63.9 \text{ Nm}. \tag{43}$$

Some of you got to this along a different route, and that's ok. Full marks.

The top of the hill is $h = 30.0$ m above the bottom of the hill. As he gets there, he stops pedalling and simply rolls down again on the other side, which slopes at the same angle as on the way up.

e) *How fast is he going, when he gets to the bottom? Assume that he started the descent with speed v (from c)), and that he is rolling without sliding all the way down.*

Solution: This is of course a question of energy conservation. There is no kinetic friction going down, and the static friction does no work. At the top, with a linear speed of v, keeping in mind the rolling without sliding, he has the kinetic energy (remembering the rotational energy of the wheels, and using the moment of inertia of the wheels)

$$E_i = \frac{1}{2}(2m+M)v^2 + 2 \times \frac{1}{2}I\omega_i^2 = \frac{1}{2}(4m+M)v^2. \tag{44}$$

The initial potential energy is $(2m+M)gh$. Since energy is conserved, we have

$$E_f = \frac{1}{2}(4m+M)v_f^2 = (2m+M)gh + \frac{1}{2}(4m+M)v^2 \tag{45}$$

$$\rightarrow v_f = \sqrt{v^2 + \frac{2gh(2m+M)}{(4m+M)}} = 24.2 \text{ m/s}. \tag{46}$$

Finally, he brakes, and the bike slides along the horizontal road. The coefficient of kinetic friction between tyre and road is $\mu_k = 1.00$.

f) *How far does he slide before coming to a halt?*

Solution: No where is kinetic friction, and energy is lost to internal energy. While sliding a distance d, the energy $\mu_k d(2m+M)g$ is lost. In order to stop, we need to lose everything it had at the top of the hill

$$\mu_k d(2m+M)g = (2m+M)gh + \frac{1}{2}(4m+M)v^2 \tag{47}$$

$$\rightarrow d = \frac{h}{\mu_k} + \frac{v^2}{2g\mu_k}\frac{4m+M}{2m+M} = 31.3 \text{ m}. \tag{48}$$

For all questions, provide an algebraic expression, as well as the numerical result.

Typical mistakes: Not making a sketch as required. Getting the forces and torques wrong. Not being able to solve the equations. Getting confused about the direction of the forces F_1, F_2 (the connections are not strings, so they don't necessarily act along the direction of the attaching sticks). Calculating the friction forces wrong. Confusing kinetic and friction forces. Kinetic friction going up and down the hill, losing energy. Forgetting the wheels in the energy. Forgetting the initial speed in the kinetic energy at the top of the hill. Adding the initial speed after having taken the square root, when computing the speed at the bottom of the hill.

Partial credit given for: Using energy conservation in e) + f) (even if done wrong). Writing down the expression for internal energy from kinetic friction. Saying something about the necessary static friction coefficient, even if forgetting to get the optimal mass distribution. Giving considerations on the mass distribution, even when calculating the friction coefficient wrong.