Høsten 2016

FYS100 Fysikk
Problems week 35

First some problems from the book:

- 3.14 (ignore the world *graphically*), 3.17, 3.18
- 3.60 (use inner product), 3.67

Solution 3.14:
This is the addition of two vectors. First from base camp to Lake A:

$$\begin{aligned}\mathbf{A} &= (280\ \text{km}, 20^\circ) = (280\cos 20^\circ\ \text{km}, 280\sin 20^\circ\ \text{km}),\\ \mathbf{B} &= (190\ \text{km}, 90^\circ + 30^\circ) = (-190\sin 30^\circ\ \text{km}, 190\cos 30^\circ\ \text{km}),\end{aligned}$$

in polar and cartesian coordinates, respectively, taking the origin as Base Camp and an East-North (x-y) coordinate system. Add them up to get

$$\mathbf{A} + \mathbf{B} = (168, 260)\ \text{km}, \tag{1}$$

or in polar coordinates

$$\mathbf{A} + \mathbf{B} = (310\ \text{km}, 57.1^\circ) \tag{2}$$

Solution 3.17:
Curious problem description. It is important to realise that the angle is relative to the road, not to the mini-van, for instance. Then the question bills down to that the camper must have a speed large enough, that it is still at least as fast as the mini-van *in the forward direction*. Since the angle of $\theta = 8.5^\circ$ is relative to forward (North), we have

$$|v_{camper}|\cos\theta \geq |v_{minivan}|. \tag{3}$$

Then all that is required is that

$$|v_{camper}| \geq \frac{|v_{minivan}|}{\cos\theta} = \frac{28\ \text{m/s}}{\cos 8.5^\circ} = 28.3\ \text{m/s}. \tag{4}$$

So when the camper moves with a speed of 28.3 m/s or more, it still has a forward component larger than 28 m/s, when pulling in at an angle of 8.5°. And the minivan doesn't need to slow down.

Solution 3.18:
This is decomposition. In polar coordinates, she walks the distance

$$\mathbf{A} = (3.10\ \text{km}, 25.0°), \tag{5}$$

with the East axis used as x, and the North axis as y. Then decomposed onto those axes to find the distance travelled when going first N, then E,

$$A_N + A_E = |\mathbf{A}| \cos\theta + |\mathbf{A}| \sin\theta = 4.12\ \text{km}. \tag{6}$$

Solution 3.60:
The request is to find the angle θ, between $\mathbf{A}$ and $\mathbf{B}$, so that

$$|\mathbf{A} + \mathbf{B}| = n|\mathbf{A} - \mathbf{B}|, \tag{7}$$

for arbitrary values of n. We square the equation and use the inner product

$$\begin{aligned} |\mathbf{A} + \mathbf{B}|^2 &= n^2|\mathbf{A} - \mathbf{B}|^2, \\ (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) &= n^2(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}), \\ |\mathbf{A}|^2 + |\mathbf{B}|^2 + 2\mathbf{A} \cdot \mathbf{B} &= n^2(|\mathbf{A}|^2 + |\mathbf{B}|^2 - 2\mathbf{A} \cdot \mathbf{B}). \end{aligned}$$

Now use the assumption that $|\mathbf{A}| = |\mathbf{B}|$, and solve for $\mathbf{A} \cdot \mathbf{B}$ to find

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|^2 \cos\theta = |\mathbf{A}|^2 \frac{n^2 - 1}{n^2 + 1} \tag{8}$$

so that

$$\cos\theta = \frac{n^2 - 1}{n^2 + 1}. \tag{9}$$

This is nicely smaller than 1; going to 1 (0°) for large n; zero (90°) for $n = 1$. Which makes sense.

Solution 3.67:
Let's build it up set by step, without thinking about number. First we start at $\mathbf{A}$. Then we go halfway to B

$$\mathbf{A} + \frac{1}{2}\mathbf{B} - \mathbf{A} = \frac{1}{2}\mathbf{B} + \mathbf{A}.$$

From there a third of the way to $\mathbf{C}$

$$\frac{1}{2}\mathbf{B} + \mathbf{A} + \frac{1}{3}(\mathbf{C} - \frac{1}{2}\mathbf{B} + \mathbf{A}) = \frac{1}{3}\mathbf{C} + \mathbf{B} + \mathbf{A}.$$

From there a quarter of the way to $\mathbf{D}$

$$\frac{1}{3}\mathbf{C}+\mathbf{B}+\mathbf{A}+\frac{1}{4}(\mathbf{D}-\frac{1}{3}\mathbf{C}+\mathbf{B}+\mathbf{A})=\frac{1}{4}\mathbf{D}+\mathbf{C}+\mathbf{B}+\mathbf{A}.$$

And finally a fifth of the way to $\mathbf{E}$

$$\frac{1}{4}\mathbf{D}+\mathbf{C}+\mathbf{B}+\mathbf{A}+\frac{1}{5}(\mathbf{E}-\frac{1}{4}\mathbf{D}+\mathbf{C}+\mathbf{B}+\mathbf{A})=\frac{1}{5}\mathbf{E}+\mathbf{D}+\mathbf{C}+\mathbf{B}+\mathbf{A}.$$

We see that the result is symmetric in $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{E}$ and so it makes no difference which tee is which, if one follows this prescription.

Additional Problem 1: Consider the vectors (in cartesian coordinates, given some basis and coordinate system),

$$\vec{\mathbf{A}}=(-1,4),\qquad \vec{\mathbf{B}}=(1,2),\qquad \vec{\mathbf{C}}=(2,1). \tag{10}$$

Compute

- The projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$.
- The projection of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{C}}$.

Is the sum of the projections equal to the original vector $\vec{\mathbf{A}}$?

Solution: The projection is given by

$$\vec{\mathbf{A}}_{\vec{\mathbf{B}}}=\frac{\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}}{\vec{\mathbf{B}}\cdot\vec{\mathbf{B}}}\vec{\mathbf{B}}. \tag{11}$$

Then we find

$$\vec{\mathbf{A}}_{\vec{\mathbf{B}}}=\frac{7}{5}(1,2)=\left(\frac{7}{5},\frac{14}{5}\right), \tag{12}$$

$$\vec{\mathbf{A}}_{\vec{\mathbf{C}}}=\frac{2}{5}(2,1)=\left(\frac{4}{5},\frac{2}{5}\right). \tag{13}$$

The sum is $\vec{\mathbf{A}}_{\vec{\mathbf{B}}}+\vec{\mathbf{A}}_{\vec{\mathbf{C}}}=(11/5,16/5)\neq(-1,4)$. This is because the vectors $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$ are not orthogonal.

Find the decomposition of $\vec{\mathbf{A}}$ onto $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$. Use whatever method you find simplest.

Solution: To find the decomposition, one may use the complicated procedure sketched at lecture. But may also just introduce numbers $c_{1,2}$ and write

$$\overrightarrow{\mathbf{A}} = c_1\overrightarrow{\mathbf{B}} + c_2\overrightarrow{\mathbf{C}} \rightarrow (-1,4) = (c_1 + 2c_2, 2c_1 + c_2) \rightarrow c_1 = 3, \quad c_2 = -2 \tag{14}$$

Additional Problem 2: Consider the vectors in 3-D

$$\overrightarrow{\mathbf{A}} = (1,2,1), \qquad \overrightarrow{\mathbf{B}} = (2,1,2). \tag{15}$$

Compute

- $\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}$. **Solution:** (3,0,-3)
- $\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}$. **Solution:** 6

Find the relative angle between the vectors, using either the scalar or the vector product. Do they agree?

Solution:

$$\cos\theta = \frac{\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}}{|\overrightarrow{\mathbf{A}}||\overrightarrow{\mathbf{B}}|} \rightarrow \theta = 35.3^\circ, \tag{16}$$

$$\sin\theta = \frac{|\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}|}{|\overrightarrow{\mathbf{A}}||\overrightarrow{\mathbf{B}}|} \rightarrow \theta = 35.3^\circ, \tag{17}$$

In fact, we have the identity

$$\left(\frac{\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}}}{|\overrightarrow{\mathbf{A}}||\overrightarrow{\mathbf{B}}|}\right)^2 + \left(\frac{|\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}|}{|\overrightarrow{\mathbf{A}}||\overrightarrow{\mathbf{B}}|}\right)^2 = \frac{(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{B}})^2 + (\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}}) \cdot (\overrightarrow{\mathbf{A}} \times \overrightarrow{\mathbf{B}})}{(\overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{A}})(\overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{B}})} = 1. \tag{18}$$