MAT100 Matematiske metoder 1 UiS

Eksamen 11.12.2015 (Egil N. Håland)

Oppgave 1

a) $z = 1 + i\sqrt{3}$

$$\underline{z + \frac{1}{z}} = 1 + i\sqrt{3} + \frac{1\cdot(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})}$$

$$= 1 + i\sqrt{3} + \frac{1}{4}(1-i\sqrt{3}) = \underline{\underline{\frac{5}{4} + i\frac{3}{4}\sqrt{3}}}$$

b) $w = \frac{1-i}{\sqrt{2}} \Rightarrow |w| = 1$ og $\mathrm{Arg}(w) = -\frac{\pi}{4}$

$$\underline{w} = e^{-i\frac{\pi}{4}} \Rightarrow w^5 = e^{-i\frac{5\pi}{4} + i2\pi} = e^{i\frac{3\pi}{4}} = \underline{\underline{\frac{-1+i}{\sqrt{2}}}}$$

c) $z^3 = -8 = 2^3 \cdot e^{i\pi} = 2^3 \cdot e^{i\pi + ik2\pi}$

$$z = 2 \cdot e^{i\frac{\pi}{3} + ik\frac{2\pi}{3}}$$

$$\underline{z_0} = 2e^{i\frac{\pi}{3}} = 2\left(\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) = \underline{\underline{1 + i\sqrt{3}}}$$

$$\underline{z_1} = 2e^{i\pi} = 2(-1+0) = \underline{\underline{-2}}$$

$$\underline{z_2} = \bar{z}_0 = \underline{\underline{1 - i\sqrt{3}}}$$

Oppgave 2

a) $$\underline{\lim_{x\to 1} \frac{1-\ln(ex)}{\sin(\pi x)}} = \lim_{x\to 1} \frac{-\frac{1}{x}}{\pi \cdot \cos \pi x} = \frac{-1}{-\pi} = \underline{\underline{\frac{1}{\pi}}}$$

b) $\lim\limits_{x\to 0} \dfrac{x^2+2x+2-2e^x}{\sin(2x)-2\cdot\sin x} = \lim\limits_{x\to 0} \dfrac{2x+2-2e^x}{2\cdot\cos 2x-2\cos x} =$

$\lim\limits_{x\to 0} \dfrac{x+1-e^x}{\cos 2x-\cos x} = \lim\limits_{x\to 0} \dfrac{1-e^x}{\sin x-2\sin 2x} =$

$\lim\limits_{x\to 0} \dfrac{-e^x}{\cos x-4\cdot\cos 2x} = \dfrac{-1}{1-4} = \underline{\underline{\dfrac{1}{3}}}$

Opgave 3

a) $\displaystyle\int_1^e \ln x\cdot x\,dx = \left[\ln x\cdot \frac{x^2}{2}\right]_1^e - \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}\,dx =$

$\displaystyle\frac{e^2}{2} - \frac{1}{2}\int_1^e x\,dx = \frac{e^2}{2} - \frac{1}{4}\left[x^2\right]_1^e = \underline{\underline{\frac{1+e^2}{4}}}$

b) $\displaystyle\int_{-3}^{-2} \frac{4\,dx}{x^2+6x+10} = \int_{-3}^{-2} \frac{4\,dx}{(x+3)^2+1} = 4\cdot\int_0^1 \frac{du}{u^2+1} =$

$\displaystyle 4\cdot\left[\tan^{-1}u\right]_0^1 = 4\cdot\left(\frac{\pi}{4}-0\right) = \underline{\underline{\pi}}$

c) $\displaystyle\int_0^1 \frac{3-x}{(x+1)\cdot(x^2+1)}\,dx$

$\displaystyle\frac{3-x}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \Big|\cdot(x+1)(x^2+1)$

$3-x \equiv (A+B)x^2+(B+C)\cdot x+(C+A) \Leftrightarrow$

$A=2 \wedge B=-2 \wedge C=1$

$\displaystyle\int_0^1 \left(\frac{2}{x+1}-\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx = \left[\ln\frac{(x+1)^2}{x^2+1}+\tan^{-1}x\right]_0^1 = \underline{\underline{\ln 2+\frac{\pi}{4}}}$

d) $\displaystyle\int_{-\pi}^{\pi} \sin(x^3)\,dx = \underline{\underline{0}}$ da $\sin(x^3)$ er odde.

Opgave 4

$$x^3 + xy^2 = 5 + y^3 + x^2y \quad \text{og} \quad P(2,1)$$

a) $VS = 8 + 2\cdot 1 = 10$
$HS = 5 + 1 + 4 = 10$ } OK

b) $3x^2 + y^2 + 2xy\cdot y' = 3y^2 y' + 2xy + x^2 y'$

$y'(2xy - x^2 - 3y^2) = 2xy - y^2 - 3x^2$

$$y' = \underline{\underline{\frac{2xy - y^2 - 3x^2}{2xy - x^2 - 3y^2}}} = \frac{4-1-12}{4-4-3} = \frac{-9}{-3} = 3$$

c) $k_T = 3 \Leftrightarrow \underline{k_N = -\frac{1}{3}}$

$N: \; y - y_1 = k_N(x - x_1) \Leftrightarrow y - 1 = -\frac{1}{3}(x-2)$

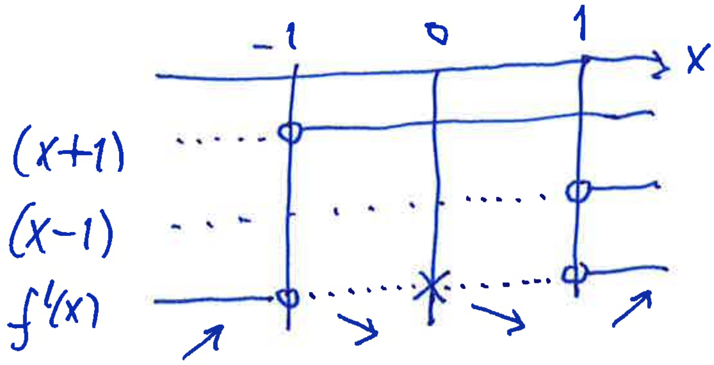
$\underline{y = -\frac{1}{3}x + \frac{5}{3}}$ eller $\underline{\underline{x + 3y = 5}}$

Opgave 5

$$f(x) = \frac{x^3}{6} + \frac{1}{2x} \quad ; \; D_f = \mathbb{R}\setminus\{0\}$$

a) $f'(x) = \frac{1}{2}x^2 - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) = \frac{1}{2}\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

$$= \frac{1}{2}\,\frac{x^2+1}{x}\cdot\frac{x^2-1}{x} = \frac{(x^2+1)(x+1)(x-1)}{2x^2}$$

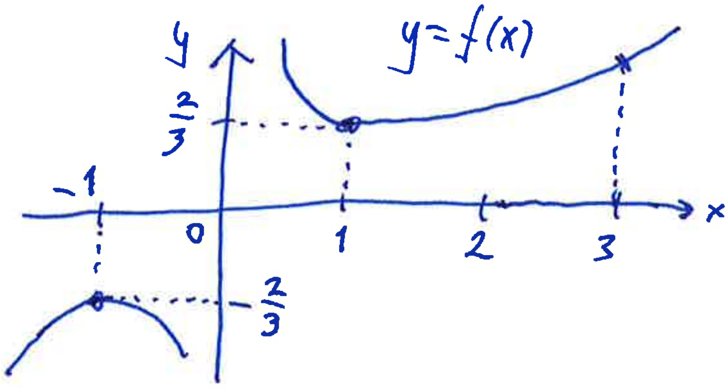


$$\underline{\underline{x_{max} = -1}} \Rightarrow \underline{\underline{y_{max}}} = f(-1) = \underline{\underline{-\frac{2}{3}}}$$

$$\underline{\underline{x_{min} = 1}} \Rightarrow \underline{\underline{y_{min}}} = f(1) = \underline{\underline{\frac{2}{3}}}$$

$$\lim_{x \to 0^+} f(x) = \infty$$

$$\lim_{x \to 0^-} f(x) = -\infty$$



b) $$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{1}{4}\left(x^2-\frac{1}{x^2}\right)^2}\,dx =$$

$$\frac{1}{2}\cdot\sqrt{4}\cdot\sqrt{1+\frac{1}{4}\left(x^2-\frac{1}{x^2}\right)^2}\,dx = \frac{1}{2}\sqrt{4+\left(x^2-\frac{1}{x^2}\right)^2}\,dx =$$

$$\frac{1}{2}\sqrt{4+x^4-2+\frac{1}{x^4}}\,dx = \frac{1}{2}\sqrt{\left(x^2+\frac{1}{x^2}\right)^2}\,dx = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)dx$$

$$s = \int ds = \frac{1}{2}\int_1^3 \left(x^2+\frac{1}{x^2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_1^3 =$$

$$\frac{1}{2}\cdot\left\{\left(9-\frac{1}{3}\right)-\left(\frac{1}{3}-1\right)\right\} = \frac{1}{2}\cdot\left(10-\frac{2}{3}\right) = 5-\frac{1}{3} =$$

$$\underline{4\frac{2}{3}} = \underline{\underline{\frac{14}{3}}}$$

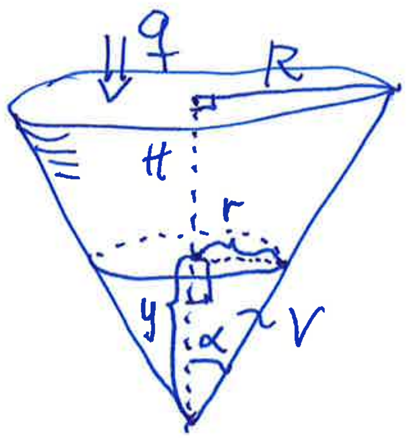
Oppgave 6

$$\frac{dy}{dx} = \frac{1+y^2}{2\sqrt{x}} \quad \Big| \cdot \frac{dx}{1+y^2}$$

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{2\sqrt{x}}$$

$$\tan^{-1} y = \sqrt{x} + C \quad \Rightarrow \quad \underline{\underline{y = \tan(\sqrt{x} + C)}}$$

Oppgave 7



$H = 4\,m$, $R = 1\,m$

$q = 35\ dm^3/min$

Finn $\dot{y}$ når $y = 3\,m = 30\,dm$

$$\tan\alpha = \frac{r}{y} = \frac{R}{H} = \frac{1}{4} \quad \Rightarrow \quad \underline{r = \frac{1}{4}y}$$

$$V = \frac{1}{3}\pi r^2 y = \frac{1}{3}\pi \cdot \frac{1}{16} y^2 \cdot y = \frac{\pi}{3 \cdot 16} \cdot y^3$$

$$\frac{dV}{dt} = q \quad \Leftrightarrow \quad \frac{\pi y^2 \cdot \dot{y}}{16} = q \quad \Leftrightarrow \quad \dot{y} = \frac{16 \cdot q}{\pi y^2}$$

$$\underline{\underline{\dot{y}}} = \frac{16 \cdot 35\ dm^3/min}{\pi \cdot 30^2 \cdot dm^2} = \frac{16 \cdot 35}{900\pi}\ dm/min = 0{,}198\ dm/min$$

$$= 1{,}98\ cm/min \approx \underline{\underline{2{,}0\ cm/min}}$$