MAT100 EKSAMEN V2016 UiS

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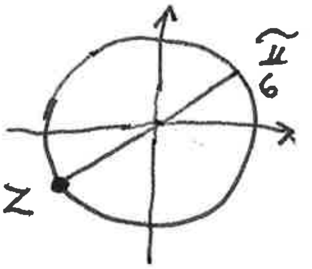
Oppgave 1

Gitt komplekst tall: $z = -\frac{\sqrt{3}+i}{2}$

a) $z^2 = \frac{(\sqrt{3}+i)^2}{4} = \frac{3+2i\sqrt{3}-1}{4} = \frac{1+i\sqrt{3}}{2}$

$|z| = \frac{|\sqrt{3}+i|}{|2|} = \frac{\sqrt{3+1}}{2} = 1$

b) $z = -\frac{1}{2}\sqrt{3} - i\frac{1}{2} \Rightarrow \tan\theta = \frac{-\frac{1}{2}}{-\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}}$



$\theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

$z = e^{-i\frac{5\pi}{6}}$

c) $z^6 = (e^{-i\frac{5\pi}{6}})^6 = e^{-i5\pi} = e^{-i5\pi + i3\cdot 2\pi}$

$= e^{i\pi} = -1$

Oppgave 2

a) $$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) = \lim_{x\to\infty}\left(\sqrt{(x+\tfrac{1}{2})^2-\tfrac{1}{4}}-x\right) =$$

$$\lim_{x\to\infty}\left(\sqrt{(x+\tfrac{1}{2})^2}-x\right) = \lim_{x\to\infty}\left(x+\tfrac{1}{2}-x\right) = \underline{\underline{\frac{1}{2}}}$$

eller :

$$\lim_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x} =$$

$$\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} =$$

$$\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \underline{\underline{\frac{1}{2}}}$$

b) $$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} \;;\; y=(\cos x)^{\frac{1}{x^2}} \Rightarrow \ln y = \frac{\ln(\cos x)}{x^2}$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x} = \lim_{x\to 0}\frac{-\sin x}{2x\cdot\cos x} =$$

$$\lim_{x\to 0}\frac{-\cos x}{2\cdot\cos x-2x\cdot\sin x} = \underline{-\frac{1}{2}}$$

$$\ln y \to -\frac{1}{2} \Rightarrow y \to e^{-\frac{1}{2}} = \underline{\underline{\frac{1}{\sqrt{e}}}}$$

Oppgave 3

a) $\int \frac{2dx}{\sqrt{1-4x^2}}$ $\quad u = 2x \Rightarrow du = 2dx$

$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \underline{\underline{\sin^{-1}(2x) + C}}$$

b) $\int (3x+1) \cdot \frac{dx}{2\sqrt{x}} = (3x+1)\cdot\sqrt{x} - \int \sqrt{x} \cdot 3dx =$

$$(3x+1)\sqrt{x} - 2x^{\frac{3}{2}} + C = \underline{\underline{(x+1)\cdot\sqrt{x} + C}}$$

c) $\int \frac{14-2x}{(x+3)(1+x^2)} dx$

$$\frac{14-2x}{(x+3)(1+x^2)} \equiv \frac{A}{x+3} + \frac{Bx+C}{1+x^2} \quad \Big| \cdot (x+3)(1+x^2)$$

$$14-2x \equiv A + Ax^2 + Bx^2 + Cx + 3Bx + 3C$$

$$14-2x \equiv (A+B)\cdot x^2 + (3B+C)\cdot x + (A+3C)$$

$$B = -A \wedge C - 3A = -2 \wedge A + 3C = 14$$

$$A = 2 \wedge B = -2 \wedge C = 4$$

$$\int \left(\frac{2}{x+3} - \frac{2x}{x^2+1} + \frac{4}{1+x^2}\right) dx = 2\ln|x+3| - \ln(x^2+1) + 4\tan^{-1}x + C$$

$$= \underline{\underline{\ln\frac{(x+3)^2}{x^2+1} + 4\cdot\tan^{-1}x + C}}$$

Oppgave 4

a) $x \cdot \frac{dy}{dx} = 3 \cdot y^{\frac{2}{3}} \quad \Big| \cdot \frac{dx}{x} \cdot y^{-\frac{2}{3}}$; $x > 0$

$$\int y^{-\frac{2}{3}} dy = 3 \cdot \int \frac{dx}{x}$$

$$3 \cdot y^{\frac{1}{3}} = 3 \cdot \ln x + 3 \cdot C \quad \Big| \cdot \frac{1}{3}$$

$$y^{\frac{1}{3}} = \ln x + C$$

$$\underline{\underline{y = (\ln x + C)^3}}$$

b) $\frac{dy}{dx} + \frac{3}{x} \cdot y = \frac{\cos x}{x^3} \quad \Big| \cdot e^{\mu} = x^3$

$$\left\{ \mu = \int \frac{3}{x} dx = 3 \ln x = \ln x^3 \right\}$$

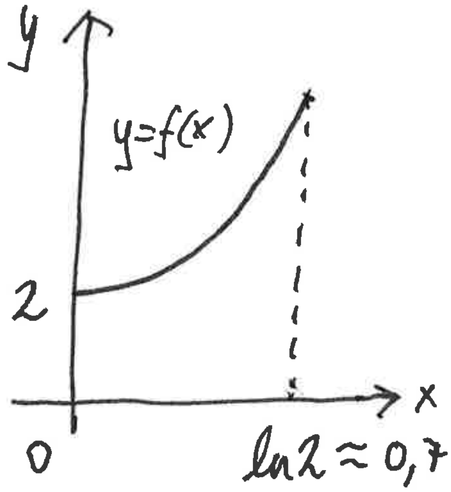
$$x^3 \cdot y' + 3x^2 \cdot y = \cos x$$

$$\frac{d}{dx}(x^3 \cdot y) = \cos x$$

$$x^3 \cdot y = \sin x + C$$

$$\underline{\underline{y = \frac{\sin x + C}{x^3}}}$$

Oppgave 5



$$f(x) = e^x + e^{-x}$$

$$D_f = [\,0,\ \ln 2\,]$$

a) $f(0) = 1+1 = \underline{\underline{2}}$

$f(\ln 2) = e^{\ln 2} + \frac{1}{e^{\ln 2}} = 2 + \frac{1}{2} = \underline{\underline{\frac{5}{2}}}$

b) $f'(x) = e^x - e^{-x}$

$f''(x) = f(x) > 0 \Rightarrow$ <u>Konkav opp.</u>

c) $V = \pi \int_0^{\ln 2} y^2 dx = \pi \int_0^{\ln 2} (e^{2x} + 2 + e^{-2x})\,dx =$

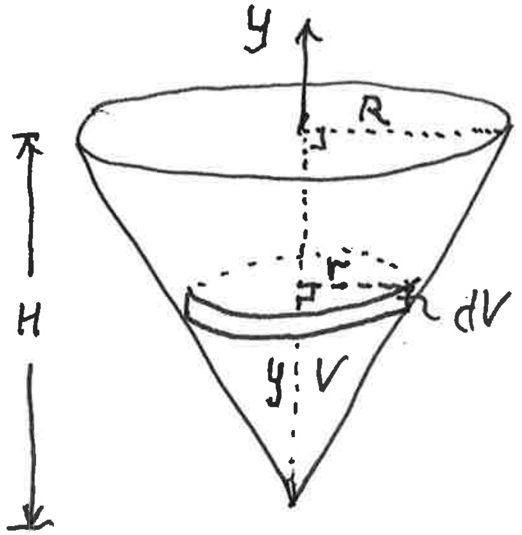
$\pi \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} \right]_0^{\ln 2} =$

$\pi \left\{ \left(\frac{1}{2}e^{\ln 4} + 2\cdot\ln 2 - \frac{1}{2}e^{-\ln 4} \right) - \left(\frac{1}{2} - \frac{1}{2} \right) \right\} =$

$\pi \cdot \left(2 + 2\cdot\ln 2 - \frac{1}{8} \right) = \underline{\underline{\pi \left(\frac{15}{8} + \ln 4 \right)}}$

$= 10,25$

Oppgave 6



$$\frac{r}{y} = \frac{R}{H} \Leftrightarrow r = \frac{R}{H} \cdot y$$

$$V_0 = \frac{1}{3}\pi R^2 H = 0{,}262 \text{ m}^3$$

$$V(y) = \frac{1}{3}\pi r^2 y = \frac{1}{3}\pi\left(\frac{R}{H}\right)^2 \cdot y^3$$

$$dV = A \cdot dy = \pi r^2 dy = \pi\left(\frac{R}{H}\right)^2 \cdot y^2 dy$$

$$H = 1{,}0 \text{ m} \;;\; R = 0{,}5 \text{ m}$$

Saltkonsentrasjon : $C(y) = \dfrac{10}{1+y^2}$ kg/m^3

Saltmengde i ΔV er $\Delta m = C(y) \cdot \Delta V$

Total saltmengde

a) $$\underline{\underline{m}} = \int dm = \int C(y)\, dV = 10\pi\left(\frac{R}{H}\right)^2 \int_0^H \frac{y^2}{1+y^2}\, dy =$$

$$10\pi \cdot \frac{1}{4} \int_0^1 \left(1 - \frac{1}{1+y^2}\right) dy = \frac{5\pi}{2}\left[y - \tan^{-1} y\right]_0^1 =$$

$$\underline{\underline{\frac{5\pi}{2} \cdot \left(1 - \frac{\pi}{4}\right) \text{ kg}}} = 1{,}685 \text{ kg}$$

b) $$\underline{\underline{\bar{C}}} = \frac{m}{V_0} = \frac{5\pi\left(1-\frac{\pi}{4}\right)}{2 \cdot \pi \cdot \frac{1}{3} \cdot \frac{1}{4}} = \frac{5\left(1-\frac{\pi}{4}\right)}{\frac{1}{6}} = \underline{\underline{30\left(1-\frac{\pi}{4}\right)}} = \underline{\underline{6{,}44 \text{ kg/m}^3}}$$

$$\underline{C_{min} = 5 \text{ kg/m}^3} \text{ og } \underline{C_{max} = 10 \text{ kg/m}^3} \;;\; \underline{\underline{C_{min} < \bar{C} < C_{max}}}$$