LØSNINGSFORSLAG MAT100 H17

① a) $z = 2-3i$, $w = 1+i$

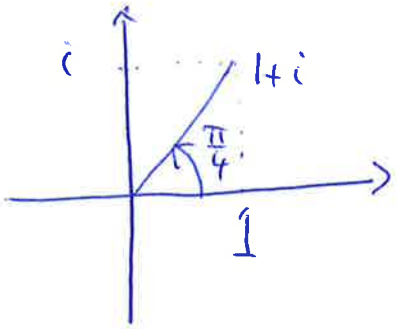
$$zw = (2-3i)(1+i) = 2-3i+2i+3 = \underline{\underline{5-i}}$$

$$|z|^2 = |2-3i|^2 = 2^2+3^2 = 4+9 = \underline{\underline{13}}$$

$$\frac{1}{w^2} = \frac{\bar{w}^2}{w^2\bar{w}^2} = \frac{(1-i)^2}{|w|^4} = \frac{1-2i+i^2}{(1^2+1^2)^2} =$$

$$= \frac{1-2i-1}{2^2} = \underline{\underline{-\frac{i}{2}}}$$

b) $(1+i)^{10}$



$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

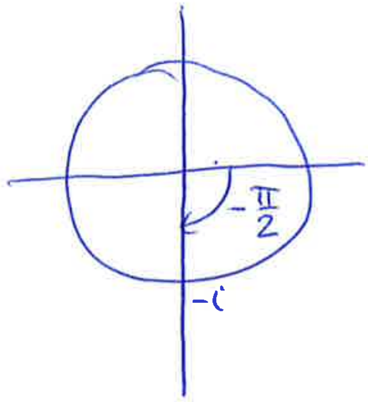
$$= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{10}$$

$$= (\sqrt{2})^{10} \cdot e^{i\frac{\pi}{4}\cdot 10}$$

$$= 2^5 \cdot e^{i\frac{5\pi}{2}} = 2^5 e^{i(2\pi+\frac{\pi}{2})} = 2^5 e^{i\frac{\pi}{2}}$$

$$= \underline{\underline{2^5 i}} = \underline{\underline{32i}}$$

c) $z^3 + i = 0 \iff z^3 = -i = e^{-\frac{\pi}{2}i}$



LØSNINGER:

$$w_1 = e^{-\frac{\pi}{2\cdot 3}i} = e^{-\frac{\pi}{6}i} = \underline{\underline{\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i}}$$

$$w_2 = e^{-\frac{\pi}{6}i + \frac{2\pi}{3}i} = e^{-\frac{\pi}{6}i + \frac{4\pi}{6}i} = e^{\frac{3\pi}{6}i} = e^{\frac{\pi}{2}i} = \underline{\underline{i}}$$

$$w_3 = e^{-\frac{\pi}{6}i + \frac{4\pi}{3}i} = e^{\frac{7\pi}{6}i} = \underline{\underline{-\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i}}$$

2 a) $\int (2x^{1/3} - e^{3x})\,dx = 2\cdot\frac{3}{4}x^{\frac{4}{3}} - \frac{1}{3}e^{3x} + C$

$$= \underline{\underline{\frac{3}{2}x^{4/3} - \frac{1}{3}e^{3x} + C}}$$

b) $\int x\cos x = x\sin x - \int 1\cdot\sin x\,dx$

$$= \underline{\underline{x\sin x + \cos x + C}}$$

c) $\int \cos x \sqrt{3-\sin x}\, dx$

$u = 3 - \sin x$

$du = -\cos x\, dx$

$= \int \sqrt{u}\,(-du)$

$= -\frac{2}{3} u^{3/2} + C = -\frac{2}{3}\sqrt{(3-\sin x)^3} + C$

$= -\frac{2}{3}(3-\sin x)^{3/2} + C$

d) $\int \frac{3x+2}{(x-1)(x^2+1)}\, dx$

DELBRØK OPPSPALTNING:

$$\frac{3x+2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$\Rightarrow 3x+2 = A(x^2+1) + (Bx+C)(x-1)$

$x=1$: $3\cdot 1+2 = A\cdot(1+1) + (B\cdot 1 + C)\cdot 0$

$5 = 2A \Rightarrow A = \frac{5}{2}$

$\underline{x=0:}$ $\quad 2 = A\cdot 1 + (B\cdot 0 + C)\cdot(-1)$

$$2 = A - C \Rightarrow C = A - 2 = \frac{5}{2} - 2 = \underline{\frac{1}{2}}$$

$\underline{x=-1:}$ $\quad 3\cdot(-1)+2 = A\cdot(1+1) + (B\cdot(-1)+C)(-1-1)$

$$-1 = 2A + 2B - 2C$$

$$-1 = 5 + 2B - 1$$

$$\underline{B = -\frac{5}{2}}$$

$$\int \frac{3x+2}{(x-1)(x^2+1)}\,dx = \int\left(\frac{\frac{5}{2}}{x-1} + \frac{-\frac{5}{2}x + \frac{1}{2}}{x^2+1}\right)dx$$

$$= \frac{5}{2}\ln|x-1| - \frac{5}{2}\int\frac{x}{x^2+1}\,dx + \frac{1}{2}\tan^{-1}x + C$$

$$\underline{\underline{= \frac{5}{2}\ln|x-1| - \frac{5}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x + C}}$$

e) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$

$$= \int (1-\sin^2 x)\cos x\,dx \qquad u = \sin x, \quad du = \cos x\,dx$$

$$= \int (1-u^2)\,du$$

$$= u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

3 a)

1) $\lim\limits_{x\to\infty} \frac{x\ln x}{x^2+1} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\ln x + \overbrace{\frac{x}{x}}^{=1}}{2x}$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

2) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+4x+5} - x\right) \qquad \text{"}\infty - \infty\text{"}$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+4x+5}-x\right)\left(\sqrt{x^2+4x+5}+x\right)}{\sqrt{x^2+4x+5}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+4x+5-x^2}{\left(\sqrt{}+x\right)}$$

$$= \lim_{x\to\infty} \frac{4x+5}{\sqrt{x^2+4x+5}+x} = \lim_{x\to\infty} \frac{(4x+5)\frac{1}{x}}{(\sqrt{}+x)\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\left(4+\frac{5}{x}\right)}{\sqrt{1+\frac{4}{x}+\frac{5}{x^2}}+1} = \frac{4}{\sqrt{1}+1} = \frac{4}{2} = \underline{\underline{2}}$$

b) $y''-3y'-10y = \cos x$

HOM. LIGN.

$y''-3y'-10y=0$

KAR. LIGN.

$r^2-3r-10=0$

$\underline{r=5, \quad r=-2}$

$\underline{y_h = Ae^{5x}+Be^{-2x}}$

PARTIKULÆR LØSNING:

$y_p = C\cos x + D\sin x$

$y_p' = -C\sin x + D\cos x$

$y_p'' = -C\cos x - D\sin x$

SETTER INN:

$$-C\cos x - D\sin x - 3(-C\sin x + D\cos x) - 10(C\cos x + D\sin x)$$

$$= (-C - 3D - 10C)\cos x + (-D + 3C - 10D)\sin x$$

$$= (-3D - 11C)\cos x + (3C - 11D)\sin x$$

$$\stackrel{!}{=} \cos x$$

$$\Rightarrow \begin{cases} \underline{3C - 11D = 0} \Rightarrow \underline{C = \frac{11}{3}D} \\ -3D - 11C = 1 \end{cases}$$

$$-3D - 11 \cdot \frac{11}{3}D = 1$$

$$\frac{-9D - 121D}{3} = 1$$

$$-\frac{130D}{3} = 1 \Rightarrow \underline{D = -\frac{3}{130}}$$

$$\underline{C = -\frac{11}{130}}$$

LØSN.: $y = y_h + y_p$

$$= Ae^{5x} + Be^{-2x} - \frac{11}{130}\cos x - \frac{3}{130}\sin x$$

c) $y\sin x = x\cos y$

$$\frac{d}{dx}(y\sin x) = \frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx}\cdot\sin x + y\cos x \overset{!}{=} \cos y - x\sin y\frac{dy}{dx}$$

$$\frac{dy}{dx}(\sin x + x\sin y) = \cos y - y\cos x$$

$$\frac{dy}{dx} = \frac{\cos y - y\cos x}{\sin x + x\sin y}$$

④ $f(x) = \dfrac{3+x}{\sqrt{9+x^2}}$, $x \in \mathbb{R}$

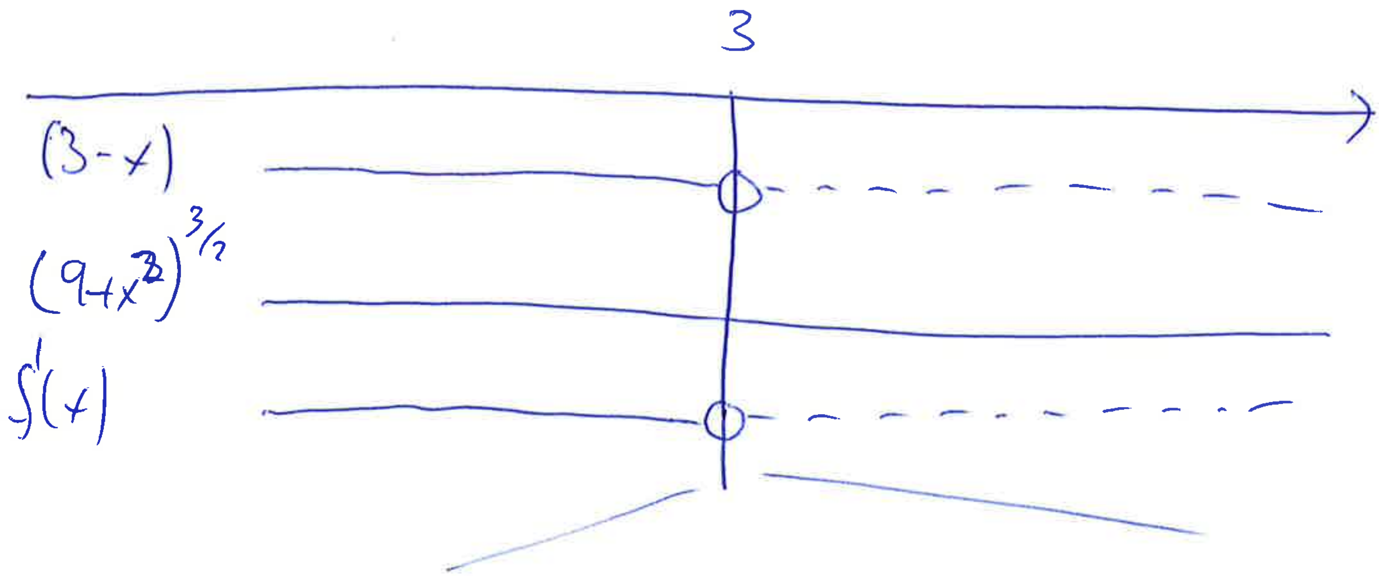
a) $f'(x) = \dfrac{1 \cdot \sqrt{9+x^2} - (3+x)\dfrac{x}{\sqrt{9+x^2}}}{(9+x^2)}$

$$= \frac{9+x^2 - 3x - x^2}{\sqrt{9+x^2}\,(9+x^2)}$$

$$= \frac{3(3-x)}{(9+x^2)^{3/2}}$$

EKSTREMALPUNKT: ENESTE MULIGHET FOR $x = 3$

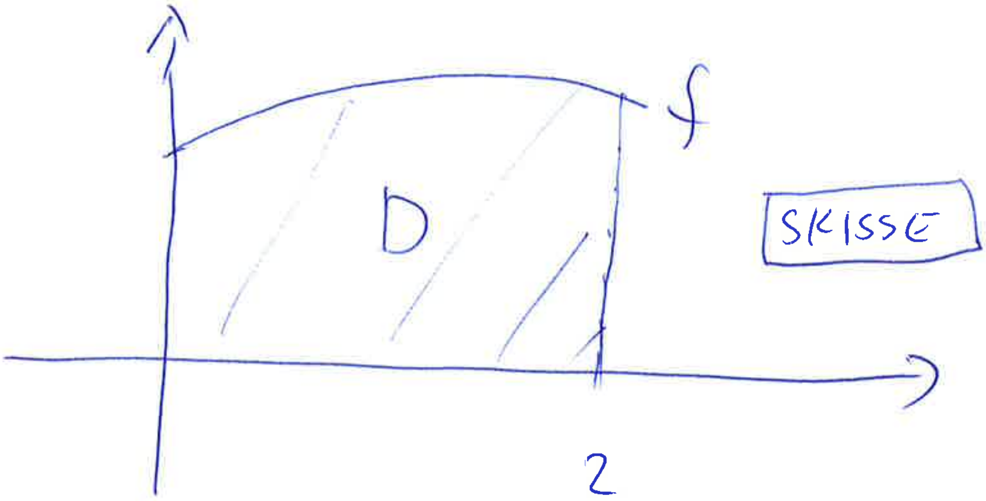
FORTEGNSSKJEMA:



MAX FOR $x = 3$

$$f(3) = \frac{3+3}{\sqrt{9+9}} = \frac{6}{3\sqrt{2}} = \sqrt{2}$$

b)



$$V = \pi \int_0^2 (f(x))^2 dx = \pi \int_0^2 \frac{(3+x)^2}{(\sqrt{9+x^2})^2} dx = \pi \int_0^2 \left(\frac{9+6x+x^2}{9+x^2}\right) dx$$

$$= \pi \int_0^2 \left(1 + \frac{6x}{9+x^2}\right) dx \quad \Big(x + 3\ln(9+x^2)\Big)\Big|_0^2$$

$$= \pi\left[\left(2 + 3\ln(9+2^2)\right) - \left(0 + 3\ln 9\right)\right]$$

$$= \pi\left[2 + 3\ln 13 - 3\cdot 2\ln 3\right]$$

$$= \underline{\underline{(2 + 3\ln 13 - 6\ln 3)\pi}}$$

(5) $\dfrac{dy}{dx} = \dfrac{y}{x}$ (SEPARABEL)

$$\frac{dy}{y} = \frac{dx}{x}$$

$\downarrow$ INT

$$\ln|y| = \ln|x| + C$$

$$|y| = |x|e^{C}$$

$$\underline{\underline{y = k \cdot x}}$$ (RETTE LINJER GJENNOM ORIGO)

C: STIGNINGSTALLET TIL C MÅ VÆRE:

$$-\frac{1}{\frac{y}{x}}$$ (NORMAL: $-\dfrac{1}{f'(x_0)}$)

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\frac{y}{x}} = -\frac{x}{y}$$ (SEPARABEL)

$$y\,dy = -x\,dx$$

$\downarrow$ INT

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C \Rightarrow \underline{y^2 + x^2 = 2C}$$

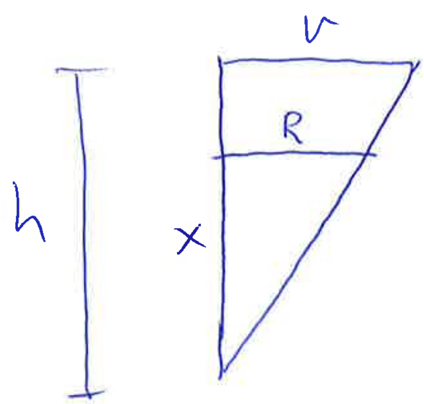
SIRKELLIGNING EVN.

DERMED MÅ C VÆRE (DELER AV) EN SIRKEL

6

TOTALT VOLUM: $V = \frac{1}{3}\pi r^2 h.$

a) FORMLIKHET:



$$\frac{r}{h} = \frac{R}{x}$$

$$R = \frac{r}{h}x = \frac{8}{10}x = \frac{4}{5}x.$$

$$V(t) = \frac{1}{3}\pi R^2 x = \frac{\pi}{3}\left(\frac{4}{5}x\right)^2 x = \frac{\pi \cdot 16}{3 \cdot 25}x^3$$

$$\frac{dV}{dt} = -k \cdot x$$

$$\frac{16\pi}{25}x^2 \cdot \frac{dx}{dt} = -k \cdot x$$

$$\Rightarrow \frac{dx}{dt} = -\frac{k \cdot 25}{16\pi x}$$

VET AT: $\left.\frac{dV}{dt}\right|_{t=0} = -8 = -k \cdot 10$

$$\Rightarrow \quad k = \frac{8}{10} = \frac{4}{5}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{4}{5} \cdot \frac{25}{16\pi x} = -\underline{\underline{\frac{5}{4\pi x}}}, \quad \dot{x}(0) = 10$$

b) LØSER ODE:

$$\frac{dx}{dt} = -\frac{5}{4\pi x} \qquad \text{(SEPARABEL)}$$

$$x\,dx = -\frac{5}{4\pi}\,dt$$

$$\frac{1}{2}x^2 = -\frac{5}{4\pi}t + C$$

$$\underline{x^2 = -\frac{5}{2\pi}t + 2C}$$

INITIALVERDI: $x(0) = 10$

$$\Rightarrow \quad 10^2 = -\frac{5}{2\pi} \cdot 0 + 2C \quad \Rightarrow \quad \underline{2C = 100}$$

$$\Rightarrow \quad x^2 = -\frac{5}{2\pi}t + 100$$

$$\underline{\underline{x = \sqrt{100 - \frac{5}{2\pi}t}}}$$

KARET ER TOMT NÅR $X = 0$

$$\Rightarrow 100 - \frac{5}{2\pi} t = 0$$

$$t = 100 \cdot \frac{2\pi}{5} = \underline{\underline{4\pi}} \simeq \underline{\underline{125{,}7}}$$

DVS. ETTER ca $\underline{\underline{126 \text{ minutter}}}$