Assignment 1

Consider the conservation law

$$u_t + f(u)_x = 0, \qquad u(x, t = 0) = u_0(x), \qquad x \in (-\infty, +\infty)$$

The method of characteristics gives a solution of the form

$$\begin{aligned} & u(x,t) = u_0(x_0) \\ \text{where} \qquad & x = x_0 + f'(u_0(x_0))t \end{aligned} \tag{1}$$

Task 1. We assume that $f(u) = \frac{1}{4}u^2$ and

$$u_0(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 1, & \frac{1}{2} < x \le 1 \\ 3 - 2x, & 1 < x \le \frac{3}{2}. \end{cases}$$

(a) Compute an expression for the exact solution $u(x,t)$ before a discontinuity (i.e., crossing characteristics) has been formed by using the method in Section 1.2. In particular, do an evaluation of characteristics emanating from different $x_0 \in [0, 3/2]$ and identify rarefaction wave and compression wave. As a part of this analysis find the breaking time T_b.

(b) Use the code "Exercise-10-Januar-2018". Modify the code such that you can solve numerically and analytically the solution at times up to breaking time T_b. Verify the analytical solution by comparing with the numerical on a fine grid (500 cells).

(c) Compute numerical solutions after the breaking time T_b and get an understanding of what happens with the solution for later times.

Task 2. We assume that $f(u) = \frac{1}{3}u^3$ and

$$u_0(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1. \end{cases}$$

(a) Compute an expression for the exact solution $u(x,t)$ at time $T_1 = 1/4$ by using the method in Section 1.2. This problem is somewhat more challenging to solve. Why?

(b) Use the code "Exercise-10-Januar-2018" again and modify the code such that you can solve numerically the solution at time $T_1 = 1/4$.

(c) Include in the code the exact solutions from (a) such that numerical and analytical solutions can be compared, similar to Exercise 10.