Problem 1.

(a) Consider the linear transport equation

$$(*) \qquad u_t + e^{-t}u_x = 0, \qquad x \in \mathbb{R} = (-\infty, +\infty)$$

with initial data

$$(**) \qquad u(x, t = 0) = \phi(x).$$

Compute the solution $u(x,t)$ and verify that it satisfies (*) and (**).

(b) Make a sketch of the characteristics in the $x - t$ coordinate system. Use this sketch and explain by words the transport process described by (*).

(c) Now we choose $\phi(x) = \exp(-x^2)$.
Make a sketch of the solution when time goes to infinity.

(d) Formulate a discrete scheme based on the upwind principle for solving the problem in (a). We consider the domain $[-L, L]$ and assume a grid of M cells with cell center $x_1, \ldots, x_M$. The scheme allows us to compute $\{u_j^{n+1}\}_{j=1}^{j=M}$ where we assume that $u_0^{n+1} = u_{M+1}^{n+1} = 0$.
Demonstrate how we can show the estimate

$$\sum_{j=1}^{M} |u_j^{n+1}| \le \sum_{j=1}^{M} |u_j^n|$$

under an appropriate condition on $\frac{\Delta t}{\Delta x}$.

(e) Compute the solution as in (a) but where we have added a source term such that we now consider

$$(*)' \qquad u_t + e^{-t}u_x = x + e^{-t}, \qquad x \in \mathbb{R} = (-\infty, +\infty)$$

Check that the new solution satisfies $(*)'$.

Problem 2.

(a) In the following we consider a horizontal 1D reservoir.
- State the single-phase porous media mass balance equation in 1D (without source term) and identify the various variables (rock and fluid).
- Introduce Darcy's law and derive an equation for the pressure where it is assumed that $\phi = \phi(p)$, $\rho = \rho(p)$, and permeability and viscosity are constant.
- Assuming a weakly compressible rock (compressibility c_r is small) we get a linear relation for $\phi(p)$.

$$\phi(p) = \phi_0[1 + c_r(p - p_0)],$$

where p_0 and ϕ_0 are reference pressure and porosity. Use this together with the assumption that the fluid is incompressible and show that we then can obtain a pressure equation of the form

$$(*) \qquad p_t = \kappa p_{xx}, \qquad x \in \mathbb{R} = (-\infty, +\infty),$$

and identify the constant parameter $\kappa > 0$.

(b) Verify that

$$(**) \qquad p(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{\kappa t}}} e^{-\theta^2}\, d\theta$$

satisfies (*).
What must be the initial condition corresponding to the solution (**) ?
(**Hint**: use that $\int_{-\infty}^{\infty} e^{-\theta^2} d\theta = \sqrt{\pi}$)

(c) Next, consider (*) with $\kappa = \frac{1}{9}$ on the spatial domain $[-1, 1]$. We assume the boundary conditions

$$p(x=-1,t) = 0, \qquad p(x=1,t) = 1.$$

Divide the domain into 6 cells with points x_1, x_2,..., x_6 located at the center of each cell. Formulate a discrete version of (*) based on an explicit time discretization for cells $j = 2, \dots, 5$.
Set

$$p_1^{n+1} = 0, \qquad p_6^{n+1} = 1$$

consistent with the boundary condition.

(d) Use the scheme defined in (c) and compute the numerical solution after a time $T = 1$ by using two time steps, i.e., $\Delta t = 1/2$ with initial data as given in (b).