Problem 1.

(a) In the following we consider a horizontal 1D reservoir.
- State the single-phase porous media mass balance equation in 1D (without source term) and identify the various variables (rock and fluid).
- Assuming a weakly compressible rock (compressibility c_r is small) we get a linear relation for $\phi(p)$.

$$\phi(p) = \phi_0[1 + c_r(p - p_0)],$$

where p_0 and ϕ_0 are reference pressure and porosity. Use this together with the assumption that the fluid is incompressible and show that we can obtain a pressure equation of the form

$$(*) \qquad p_t = \varepsilon p_{xx}, \qquad x \in \mathbb{R} = (-\infty, +\infty),$$

and identify the constant parameter $\varepsilon > 0$.

(b) Setting $\varepsilon = 1$ in (*) we know that

$$(**) \qquad p(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$$

satisfies (*) with initial data equal to Heaviside function

$$p(x, t = 0) = \begin{cases} 0, & x < 0; \\ 1, & x > 0. \end{cases}$$

- Make use of (**) combined with an appropriate rescaling of x and derive an expression for the solution of (*) with $\varepsilon > 0$.
- Sketch the solution for a fixed time T and two different values of ε in order to indicate the impact from ε on the solution.

(c) We now consider the pressure equation (*) on the domain $x \in (-1, 1)$. In addition, we introduce a source term of the form $-K(p - p^*)$ where p^* is a known, constant pressure

$$(***) \qquad \begin{aligned} & p_t = \varepsilon p_{xx} - K(p - p^*), \qquad x \in (-1, 1), \qquad K > 0 \text{ (constant)} \\ & p_x(-1, t) = p_x(+1, t) = 0. \end{aligned}$$

Demonstrate how we can derive a stability estimate for the pressure p in (***) in terms of an estimate of $\int_0^1 (p - p^*)^2 dx$. What does this stability estimate tell us?

(d) Set $\varepsilon = 2/5$ and $K = p^* = 1$ and introduce a discrete scheme for (***). Consider an initial pressure $p_0(x)$

$$p_0(x) = \begin{cases} -x, & x < 0; \\ +x, & x \geq 0. \end{cases}$$

Consider a grid of 5 cells on the domain $x \in (-1, 1)$ corresponding to

$$x_1 = -\frac{4}{5}, \quad x_2 = -\frac{2}{5}, \quad x_3 = 0, \quad x_4 = \frac{2}{5}, \quad x_5 = \frac{4}{5}.$$

Make use of the discrete scheme and compute a numerical solution after 1 time step where $\Delta t = 1/5$. Check stability condition. Try to give a brief physical interpretation of the resulting pressure solution.

Problem 2.

(a) Consider the linear transport equation

$$(*) \qquad u_t + \Big(\frac{x}{2-t}\Big)u_x = q(x,t,u), \qquad x \in \mathbb{R} = (-\infty, +\infty)$$

with initial data

$$(**) \qquad u(x, t = 0) = \phi(x).$$

Set $q(x,t,u) = 0$.
- For what time interval is the model well-defined?
- Compute the solution $u(x,t)$ by using the method of characteristics.
- Make a sketch of characteristics and explain by words the transport process reflected by a characteristic.
- Verify by direct computation that your solution $u(x,t)$ satisfies (*) and (**).

(b) Consider (*) with $q(x,t,u) = x$.
- Compute the solution $u(x,t)$ by using the method of characteristics. Verify that your solution satisfies (*) and (**)

(c) Consider the solution in (a) with initial data $\phi(x) = 1 - x^2$.
- Make a sketch of the solution in the interval $[-2, +2]$ at time $t = 0$ and $t = 1$.
- What happens with the solution as $t \to 2$?
- Describe briefly by words the transport process represented by this solution.

(d) Consider the solution in (b) with initial data $\phi(x) = 1 - x^2$.
- Make a sketch of the solution in the interval $[-2, +2]$ at time $t = 0$ and $t = 1$.
- What happens with the solution as $t \to 2$?
- Describe briefly by words the transport process represented by this solution.

(e) Now, consider the transport equation

$$u_t + \frac{1}{2}u_x = 0, \qquad x \in [0, 1]$$

with initial data and boundary data

$$u(x, t = 0) = 0, \qquad u(x = 0, t) = 1.$$

- Describe the characteristics for this model and make a plot of some of them for $x \in [0, 1]$. Compute the general solution and make a sketch of the solution $u(x, t = 1/2)$.

(f) Consider a scheme for the model in (e) given in the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2\Delta x}(U_{j+1/2}^n - U_{j-1/2}^n) = 0.$$

based on upwind discretization. Use it to compute numerical solutions for a grid of 6 cells with cell centers x_1, x_2, ... , x_6. Compute the solution at time $t = 1/2$ by using 2 timesteps. For the first cell, set $U_{1/2} = 1$ to take into account the left boundary condition. Make a figure where you include exact solution and numerical approximate solution computed from the scheme.