Problem 1: Solution.

(a) In the following we consider a horizontal 1D reservoir.
- State the single-phase porous media mass balance equation in 1D (without source term) and identify the various variables (rock and fluid).
- Assuming a weakly compressible rock (compressibility c_r is small) we get a linear relation for $\phi(p)$.

$$\phi(p) = \phi_0[1 + c_r(p - p_0)],$$

where p_0 and ϕ_0 are reference pressure and porosity. Use this together with the assumption that the fluid is incompressible and show that we can obtain a pressure equation of the form

$$(*) \qquad p_t = \varepsilon p_{xx}, \qquad x \in \mathbb{R} = (-\infty, +\infty),$$

and identify the constant parameter $\varepsilon > 0$.

Solution:

Mass balance

$$(\phi\rho)_t + (\rho u)_x = 0,$$

where ϕ, ρ, and u are porosity, fluid density, and fluid velocity (Darcy velocity).

Darcy's law:

$$u = -\frac{k}{\mu} p_x$$

This gives

$$(\phi(p)\rho(p))_t = (\frac{k}{\mu}\rho(p)p_x) = \frac{k}{\mu}(\rho(p)p_x)_x$$

Using assumptions on ϕ and ρ we get

$$\rho\phi_0[1 + c_r(p - p_0)]_t = \rho\phi_0 c_r p_t = \frac{k}{\mu}\rho p_{xx}.$$

This gives us

$$p_t = \varepsilon p_{xx}, \qquad \varepsilon = \frac{k}{\mu\phi_0 c_r}$$

(b) Setting $\varepsilon = 1$ in (*) we know that

$$(**) \qquad p(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{t}}} e^{-\theta^2} d\theta$$

satisfies (*) with initial data equal to Heaviside function

$$p(x, t = 0) = \begin{cases} 0, & x < 0; \\ 1, & x > 0. \end{cases}$$

- Make use of (**) combined with an appropriate rescaling of x and derive an expression for the solution of (*) with $\varepsilon > 0$.
- Sketch the solution for a fixed time T and two different values of ε in order to indicate the impact from ε on the solution.

Solution:

Introduce $\hat{x} = x/\sqrt{\varepsilon}$ and consider

$$p(x,t) = p(\hat{x}\sqrt{\varepsilon}, t) = \hat{p}(\hat{x}, t)$$

Then it follows that

$$\frac{\partial p}{\partial x} = \frac{\partial \hat{p}}{\partial x} = \frac{\partial \hat{p}}{\partial \hat{x}} \cdot \frac{\partial \hat{x}}{\partial x} = \frac{\partial \hat{p}}{\partial \hat{x}} \frac{1}{\sqrt{\varepsilon}}, \qquad \frac{\partial^2 p}{\partial x^2} = \hat{p}_{\hat{x}\hat{x}} \frac{1}{\varepsilon}, \qquad p_t = \hat{p}_t$$

Hence,

$$\hat{p}_t = \hat{p}_{\hat{x}\hat{x}}$$

Since initial data $\hat{p}(x, t = 0)$ is the same as $p(x, t = 0)$, we can conclude that

$$\hat{p}(\hat{x}, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{\hat{x}}{2\sqrt{t}}} e^{-\theta^2} d\theta = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\sqrt{\varepsilon t}}} e^{-\theta^2} d\theta = p(x, t)$$

The effect of ε, can be seen from the above solution is a smearing out effect on the initial jump which increases with larger ε.

(c) We now consider the pressure equation (*) on the domain $x \in (-1, 1)$. In addition, we introduce a source term of the form $-K(p - p^*)$ where p^* is a known, constant pressure

$$(***) \qquad p_t = \varepsilon p_{xx} - K(p - p^*), \qquad x \in (-1, 1), \qquad K > 0 \text{ (constant)}$$
$$p_x(-1, t) = p_x(+1, t) = 0.$$

Demonstrate how we can derive a stability estimate for the pressure p in (***) in terms of an estimate of $\int_0^1 (p - p^*)^2 dx$.
Express briefly what this stability estimate tells us?

Solution:

Introduce $P = p - p^*$ and write model as

$$P_t = \varepsilon P_{xx} - KP.$$

Multiply by P and integrate over $[-1, 1]$ to get

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} P^2 dx = \varepsilon \int_{-1}^{1} P_{xx} P dx - K \int_{-1}^{1} P^2 dx = \varepsilon \int_{-1}^{1} (P_x P)_x dx - \varepsilon \int_{-1}^{1} P_x^2 - K \int_{-1}^{1} P^2 dx \leq 0$$

by using boundary condition and negative sign of terms. This gives

$$\int_{-1}^{1} P^2(x, t) dx \leq \int_{-1}^{1} P(x, t = 0)^2 dx$$

(d) Set $\varepsilon = 2/5$ and $K = p^* = 1$ and introduce a discrete scheme for (***). Consider an initial pressure $p_0(x)$

$$p_0(x) = \begin{cases} -x, & x < 0; \\ +x, & x \geq 0. \end{cases}$$

Consider a grid of 5 cells on the domain $x \in (-1, 1)$ corresponding to

$$x_1 = -\frac{4}{5}, \quad x_2 = -\frac{2}{5}, \quad x_3 = 0, \quad x_4 = \frac{2}{5}, \quad x_5 = \frac{4}{5}.$$

Make use of the discrete scheme and compute a numerical solution after 1 time steps where $\Delta t = 1/5$. Try to give a brief physical interpretation of the resulting pressure solution.

Solution:
Model:

$$p_t = \frac{2}{5}p_{xx} - (p-1).$$

Scheme: We have that $\varepsilon\frac{\Delta t}{\Delta x^2} = (2/5)\frac{1/5}{(2/5)^2} = 1/2$. Hence, stability condition $\varepsilon\frac{\Delta t}{\Delta x^2} \le 1/2$ is ok.
General Scheme:

$$\begin{aligned} p_j^{n+1} &= p_j^n + \varepsilon\frac{\Delta t}{\Delta x^2}(p_x|_{j+1/2} - p_x|_{j-1/2}) - \Delta t(p_j^n - 1) \\ &= p_j^n + \frac{1}{2}(p_x|_{j+1/2} - p_x|_{j-1/2}) - \frac{1}{5}(p_j^n - 1) \end{aligned}$$

Inserting numbers we get

$$\begin{aligned} p_1^1 &= p_1^0 + \frac{1}{2}([p_2^0 - p_1^0] - 0) - \frac{1}{5}(p_1^0 - 1) = 4/5 - 1/2 * 2/5 - \frac{1}{5}(4/5 - 1) = 4/5 - 4/25 = 0.64 \\ p_j^1 &= p_j^0 + \frac{1}{2}([p_{j+1}^0 - p_j^0] - [p_j^0 - p_{j-1}^0]) - \frac{1}{5}(p_j^0 - 1) = 1/2(p_{j-1}^0 + p_{j+1}^0) - \frac{1}{5}(p_j^0 - 1), \quad j = 2, 3, 4 \\ p_5^1 &= p_5^0 + \frac{1}{2}(0 - [p_5^0 - p_4^0]) - \frac{1}{5}(p_5^0 - 1) = 4/5 - 1/2 * 2/5 - \frac{1}{5}(4/5 - 1) = 4/5 - 4/25 = 0.64 \end{aligned}$$

Then we get $p_j^0 = (4/5, 2/5, 0, 2/5, 4/5)$.
Time t^1:

$$p_j^1 = \begin{pmatrix} \frac{16}{25} & \frac{13}{25} & \frac{15}{25} & \frac{13}{25} & \frac{16}{25} \end{pmatrix} = \begin{pmatrix} 0.64 & 0.52 & 0.6 & 0.52 & 0.64 \end{pmatrix}$$

The solution reflects that there is competition between diffusion (smearing out initial pressure profile), and increase of pressure through the source term sine $-(p-1)$ gives a positive contribution.

Problem 2: Solution.

(a) Consider the linear transport equation

$$(*) \qquad u_t + \left(\frac{x}{2-t}\right)u_x = q(x,t,u), \qquad x \in \mathbb{R} = (-\infty, +\infty)$$

with initial data

$$(**) \qquad u(x, t=0) = \phi(x).$$

Set $q(x,t,u) = 0$.

Solution:
Well-defined for $t \in [0, 2)$. Velocity $\frac{x}{2-t}$ blows up as $t \to 2^-$.
Characteristic:

$$\frac{dx}{dt} = \frac{x}{2-t}, \qquad x(t=0) = x_0$$

which implies that $x(t) = x_0\frac{2}{2-t}$. Plotting in x-t diagram, we get a path that starts at $x_0 > 0$ for $t = 0$ and is bending towards $+\infty$ as $t \to 2^-$. Similarly, for $x_0 < 0$ the path will go to $-\infty$ as $t \to 2^-$.

Solution $u(x,t)$:

$$u(x,t) = \phi(x_0) = \phi\Big(x\frac{2-t}{2}\Big)$$

Check:

– Firstly, $u(x,t=0) = \phi(x)$.

– Secondly,

$$u_t = \phi'(\cdot)\frac{x}{2}(-1), \qquad u_x = \phi'(\cdot)\frac{2-t}{2}$$

Thus,

$$u_t + \frac{x}{2-t}u_x = 0.$$

(b) Consider (*) with $q(x,t,u) = x$.
- Compute the solution $u(x,t)$ by using the method of characteristics. Verify that your solution satisfies (*) and (**)

Solution:

Solution $u(x,t)$:

$$\frac{du(x(t),t)}{dt} = x(t) = x_0\frac{2}{2-t}$$

which gives after integration over $[0,t]$

$$u(x,t) - \phi(x_0) = -2x_0 \ln\Big(\frac{2-t}{2}\Big)$$

resulting in

$$u(x,t) = \phi\Big(x\frac{2-t}{2}\Big) - x(2-t)\ln\Big(\frac{2-t}{2}\Big)$$

Check:

Firstly, $u(x,t=0) = \phi(x)$.

Secondly,

$$u_t = \phi'(\cdot)\frac{x}{2}(-1) + x\ln\Big(\frac{2-t}{2}\Big) + x(2-t)\frac{1}{2-t}, \qquad u_x = \phi'(\cdot)\frac{2-t}{2} - (2-t)\ln\Big(\frac{2-t}{2}\Big)$$

Thus,

$$u_t + \frac{x}{2-t}u_x = x.$$

(c) Consider the solution in (a) with initial data $\phi(x) = 1 - x^2$.

Solution:

$$u(x,t) = 1 - x^2(1-\frac{t}{2})^2$$

As $t \to 2^-$, $u(x,t) \to 1$.

(d) Consider the solution in (b) with initial data $\phi(x) = 1 - x^2$.

Solution:

$$u(x,t) = 1 - x^2(1-\frac{t}{2})^2 - x(2-t)\ln\Big(\frac{2-t}{2}\Big)$$

As $t \to 2^-$, $u(x,t) \to 1$.

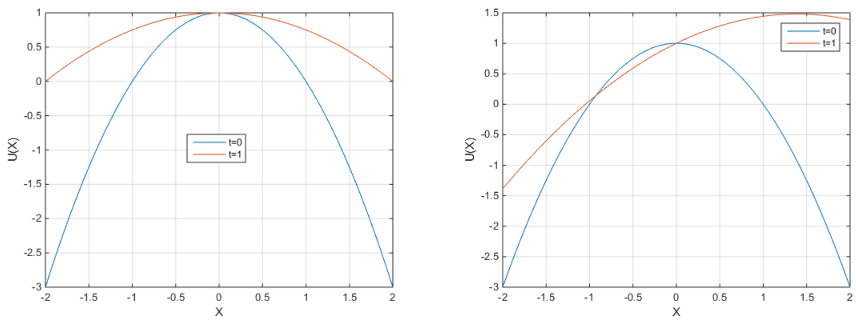

FIGURE 1. Left: solution without source term. Right: solution with source term.

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(e) Now, consider the simpler transport equation

$$u_t + \frac{1}{2}u_x = 0, \qquad x \in [0,1]$$

with initial data

$$u(x, t=0) = 0,$$

and boundary data

$$u(x=0, t) = 1.$$

- Describe the characteristics for this model and make a plot of some of them for $x \in [0,1]$. Make a sketch of the solution $u(x, t = 1/2)$.

Solution:

Characteristics:

$$x(t) = \frac{1}{2}t + x_0, \qquad u(x,t) = \begin{cases} 1, & \text{for } x \le \frac{1}{2}t; \\ 0, & \text{for } x > \frac{1}{2}t \end{cases}$$

and

$$u(x, t = \frac{1}{2}) = \begin{cases} 1, & \text{for } x \le \frac{1}{4}; \\ 0, & \text{for } x > \frac{1}{4} \end{cases}$$

(e) Present a scheme for the model in (e) given in the form

$$(S) \qquad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2\Delta x}(U_{j+1/2}^n - U_{j-1/2}^n) = 0.$$

based on upwind discretization discretization. Use it to compute numerical solutions for a grid of 6 cells with cell centers x_1, x_2, ... , x_6. Compute the solution at time $t = 1/2$ by using 2 timesteps. For the first cell, set $U_{1/2} = 1$ to take into account the left boundary condition.

Solution:
Scheme: We have that $\frac{\Delta t}{2\Delta x} = \frac{3}{4}$.

$$u_1^{n+1} = u_1^n - \frac{3}{4}(u_1^n - 1) = \frac{1}{4}u_1^n + \frac{3}{4}$$

$$u_j^{n+1} = u_j^n - \frac{3}{4}(u_j^n - u_{j-1}^n) = \frac{1}{4}u_j^n + \frac{3}{4}u_{j-1}^n, \qquad j = 2, 3, ..., 6$$

We have that $u_j^0 = 0$ for $j = 1, \ldots, 6$.
Time t^1:

$$u_j^1 = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Time t^2:

$$u_j^2 = \begin{pmatrix} \frac{15}{16} & \frac{9}{16} & 0 & 0 & 0 & 0 \end{pmatrix}$$